Separation and Homology

Semester Report

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Supervision: Dr. Milagros Izquierdo Barrios

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Olaf Beyersdorff

Contents

0 Introduction

The main subject of this paper is the Jordan-Brouwer separation theorem. It is this one of those theorems in mathematics which strongly appeal to our intuition but possess a very complicated proof, often using sophisticated techniques. The theorem originated in 1893 with Jordan and was later generalized by Brouwer, both of whom used fairly difficult geometric arguments for the proof. It turns out however that homology theory as developed at around the same time as the works of Jordan and Brouwer is best used to prove the Jordan-Brouwer theorem. Therefore this paper is devided into two parts the first of which gives an introduction to homology theory. The intention was to provide as much of homology theory as necessary to prove the Jordan-Brouwer separation theorem which is together with other results by Brouwer as the fixed-point theorem an important application of homology. It is for this reason that some material that should have been present in a survey of homology theory is omitted. Nevertheless all basic concepts of singular homology are rigorously developed and nearly all the proofs are given. For further material and more geometric explanations the reader is referred to the reference list, especially Massey, Rotman and Stöcker/Zieschang shall be recommended.

Part II deals with the Jordan-Brouwer separation theorem and related subjects. In Section 7 a complete proof for the Jordan-Brouwer theorem is given, first for the sphere S^n from which we then deduce the version for $\mathbf{R}^n$. As a corollary the invariance of domain is proved and the Phragmen-Brouwer separation properties are dicussed. Section 8 then deals with the Schönflies conjecture, a question that comes out quite naturally from the Jordan-Brouwer theorem in Section 7. Various forms of the Schönflies theorem for different dimensions as well as counterexamples such as the Alexander horned sphere and Antoine's set are examined. It is actually in this section that we leave the domain of homology theory. This and the fact, that the proofs, which use sophisticated arguments from geometric and point-set topology, are quite long and would cover several pages, are the reasons why most of the results are merely stated here. For more details one should consult the books by Christenson/Voxman, Moise and Rourke/Sanderson. The last chapter gives a detailed account of the history and the development of the theorems by Jordan, Brouwer and Schönflies and related topics as dicussed in Sections 7 and 8.

It shall also be said that we could have chosen a different way to develop homology theory in Part I using simplicial complexes, CW-complexes or singular cubes instead of singular simplexes. Singular homology is in some respect the most general approach to homology since we can use arbitrary topological spaces instead of triangulable spaces or CW-complexes. Although these different methods turn out to be equivalent to each other they have different advantages. Using singular homology some definitions are simplified. However it is quite complicated to determine the homology groups for given spaces. With simplicial complexes geometric interpretation of the basic concepts becomes easier. Computation of homology groups is sometimes lenghty but more elementary. For these reasons and also because we will need the notion of a simplicial complex in Section 8 we shall just outline some of the basic definitions of the simplicial theory in the following.

Let $x_0, \ldots, x_q \in \mathbf{R}^n$ be $q+1$ points in $\mathbf{R}^n$, $q \leq n$, which are not contained in any $(q-1)$-dimensional hyperplane of $\mathbf{R}^n$. Then the *q-dimensional simplex*

$$\sigma = \sigma_q = x_0 x_1 \ldots x_q$$

is the convex hull of the points $x_0, \ldots, x_q$. We call $x_0, \ldots, x_q$ the vertices of σ_q. A simplex τ is a face of σ if all vertices of τ are also vertices of σ. We write this as $\tau < \sigma$. A 1-dimensional simplex is called an edge.

A *simplicial complex* is a collection K of simplexes in $\mathbf{R}^n$ with the following properties

1. If $\sigma \in K$ and $\tau < \sigma$ then also $\tau \in K$.
2. If $\sigma, \tau \in K$ and $\sigma \cap \tau \neq \emptyset$ then $\sigma \cap \tau < \sigma$ and $\sigma \cap \tau < \tau$.

If K is a simplicial complex then $|K|$ is the union of all the simplexes of K with the subspace topology induced by the topology of $\mathbf{R}^n$. We call $|K|$ a polyhedron. We may then proceed to define the chain groups, the boundary operator ∂ and the homology groups in a way analogous to the approach in Section 1.

A topological space X is called *triangulable* if there is a complex K such that $|K| \approx X$. It can be shown that a large class of spaces is triangulable, for instance every compact, differentiable manifold can be triangulated. For the purpose of proving the Jordan-Brouwer separation theorem in its full generality, however, it is not sufficient to limit our attention to triangulable spaces. Therefore we will start to develop singular homology theory for arbitrary topological spaces in the next section.

0.1 Notation

We list some terminology and notations that will be used frequently in the following sections.

$\mathbf{Z}$	ring of integers
$\mathbf{Z}^n$	set of all n-tupels $(x_1, \ldots, x_n)$
$\mathbf{Z}_k$	integers modulo k
$\mathbf{R}^n$	Euclidean n-space
$D^n = \{x \in \mathbf{R}^n : \lvert x \rvert \leq 1\}$	n-dimensional disc or ball
$S^n = \{x \in \mathbf{R}^n : \lvert x \rvert < 1\}$	n-dimensional sphere
$I = [0, 1]$	unit interval

A set which is homeomorphic to D^n (or I^n) is called an n-ball. If A is a set then A°, $\dot{A}$ and $\bar{A}$ denote the interiour, the boundary and the closure of A, respectively. Moreover we shall use the relations

$$\begin{array}{ll} \cong & \text{isomorphic} \\ \approx & \text{homeomorphic} \\ \simeq & \text{homotopic.} \end{array}$$

Part I
Homology Theory

1 Definition of Homology Groups

1.1 The Singular Complex of a Space

Definition 1.1 Let $q \geq 0$. We call the points

$$e_0 = (1, 0, \ldots, 0) \quad e_1 = (0, 1, 0, \ldots, 0) \quad \ldots \quad e_q = (0, \ldots, 0, 1)$$

the unit points of $\mathbf{R}^{q+1}$. We define the standard q-simplex Δ_q to be the following subset of $\mathbf{R}^{q+1}$:

$$\Delta_q = \{x \in \mathbf{R}^{q+1} : x = \sum_{i=0}^{q} \lambda_i e_i \text{ with } 0 \leq \lambda_i \leq 1 \text{ and } \sum_{i=0}^{q} \lambda_i = 1\}$$

The points e_0, ..., e_q are called the vertices of Δ_q.

For instance, Δ_0 is a single point, Δ_1 is a segment, Δ_2 an equiliteral triangle and Δ_3 is a regular tetrahedron.

Definition 1.2 A mapping $f : \Delta_q \to \mathbf{R}^n$ is called linear if there exists a linear map $F : \mathbf{R}^{q+1} \to \mathbf{R}^n$ such that $F|_{\Delta_q} = f$. For arbitrary points $x_0, \ldots, x_q \in \mathbf{R}^n$ there is a unique linear map $f : \Delta_q \to \mathbf{R}^n$ such that $f(e_i) = x_i$ for $i = 0, \ldots, q$, namely $f(x) = \sum_{i=0}^{q} \lambda_i x_i$. Thus a linear map of Δ_q is completely determined by its values on the vertices of Δ_q. For $q \geq 1$ and $0 \leq j \leq q$ consider the linear map $\delta^j_{q-1} : \Delta_{q-1} \to \Delta_q$ induced by

$$\begin{aligned} \delta^j_{q-1}(e_i) &= e_i && \text{for } i < j \\ \delta^j_{q-1}(e_i) &= e_{i+1} && \text{for } i \geq j. \end{aligned}$$

The image of δ^j_{q-1} is called the j-th face of Δ_q. It consists of all points $(\lambda_0, \ldots, \lambda_q) \in \Delta_q$ with $\lambda_j = 0$. The union of all faces of Δ_q is called the boundary of Δ_q which we denote by $\dot{\Delta}_q$.

Lemma 1.3 *For $q \geq 2$ and $0 \leq k < j \leq q$ we have $\delta^j_{q-1}\delta^k_{q-2} = \delta^k_{q-1}\delta^{j-1}_{q-2}$.*

Proof. Both sides map the vertices of Δ_{q-2} as follows:

$$\begin{aligned} e_i &\mapsto e_i && \text{for } i < k \\ e_i &\mapsto e_{i+1} && \text{for } k \leq i \leq j-1 \\ e_i &\mapsto e_{i+2} && \text{for } i \geq j-1. \end{aligned}$$

From this and from linearity we conclude that the maps are equal. $\square$

Definition 1.4 Let X be a topological space. A *singular q-simplex* of X is a continous map $\sigma = \sigma_q : \Delta_q \to X$. For each $q \geq 0$ define $S_q(X)$ as the free abelian group with basis all singular q-simplexes in X. The elements of S_q are called *singular q-chains* in X. Hence every $c \in S_q(X)$ has a unique representation as a finite linear combination $c = \sum_\sigma n_\sigma \sigma$, with coefficients $n_\sigma \in \mathbf{Z}$. For $q < 0$ we set $S_q(X) = 0$.

For $q \geq 1$ we define a homomorphism $\partial = \partial_q : S_q(X) \to S_{q-1}(X)$, called the *boundary operator*

$$\partial_q(\sigma) = \sum_{i=0}^{q} (-1)^i (\sigma \delta^i_{q-1})$$

For $q \leq 0$ put $\partial_q = 0$.

We have constructed a sequence of free abelian groups and homomorphisms

$$\cdots \longrightarrow S_{q+1}(X) \xrightarrow{\partial_{q+1}} S_q(X) \xrightarrow{\partial_q} S_{q-1}(X) \longrightarrow \cdots$$

We call this sequence the *singular complex* of X.

Theorem 1.5 *For all q we have*

$$\partial_{q-1}\partial_q = 0.$$

Proof. Since $S_q(X)$ is generated by all q-simplexes σ it suffices to show $\partial\partial\sigma = 0$ for every σ. Using Lemma 1.3 we get:

$$\begin{aligned}
\partial_{q-1}\partial_q\sigma &= \partial_q(\sum_{j=0}^{q} (-1)^j \sigma\delta^j_{q-1}) \\
&= \sum_{j=0}^{q} (-1)^j (\sum_{k=0}^{q-1} (-1)^k \sigma\delta^j_{q-1}\delta^k_{q-2}) \\
&= \sum_{j \leq k} (-1)^{j+k} \sigma\delta^j_{q-1}\delta^k_{q-2} + \sum_{k<j} (-1)^{j+k} \sigma\delta^k_{q-1}\delta^{j-1}_{q-2}.
\end{aligned}$$

In the second sum change variables: replace k by j and j by $k+1$. We get $\sum_{j \leq k} (-1)^{j+k+1} \sigma\delta^j_{q-1}\delta^k_{q-2}$. Terms cancel and we get $\partial\partial\sigma = 0$. $\square$

Definition 1.6 We now define the following groups

$$\begin{aligned}
Z_q(X) &= \ker \partial_q \\
B_q(X) &= \operatorname{im} \partial_{q+1}
\end{aligned}$$

We call $Z_q(X)$ the group of *singular q-cycles* in X and $B_q(X)$ the group of *singular q-boundaries* in X. Clearly, $Z_q(X)$ and $B_q(X)$ are subgroups of $S_q(X)$ and since $\partial_q\partial_{q+1} = 0$ we get

$$B_q(X) \subseteq Z_q(X).$$

Hence we can define

$$H_q(X) = Z_q(X)/B_q(X).$$

$H_q(X)$ is called the *q-dimensional singular homology group* of X. The elements of $H_q(X)$ are the cosets $\{z\} = \{z\}_X = z + B_q(X)$ with $z \in Z_q(X)$. We call $\{z\}$ the homology class of z.

1.2 Reduced Homology Groups

In the definition of the boundary operator we deliberatly chose to define $\partial_0 = 0$. However, we can also use a different homomorphism $\varepsilon : S_0(X) \to \mathbf{Z}$ which is called the augmentation. Let $c = \sum_\sigma n_\sigma \sigma$. Then we define

$$\varepsilon(c) = \varepsilon(\sum_\sigma n_\sigma \sigma) = \sum_\sigma n_\sigma$$

We easily get the formula

$$\varepsilon\partial_1 = 0.$$

To prove this it suffices to show $\varepsilon\partial_1(\sigma)$ for every 1-dimensional simplex σ in X, but this is trivial. Now we define $\tilde{Z}_0(X) = \ker \varepsilon$. Because of $\varepsilon\partial_1 = 0$ we get $B_0(X) \subseteq \tilde{Z}_0(X)$ and can therefore form the quotient

$$\tilde{H}_0(X) = \tilde{Z}_0(X)/B_0(X)$$

It is convenient to let $\tilde{H}_q(X) = H_q(X)$ for $q > 0$. The groups $\tilde{H}_q(X)$ are called the *reduced q-dimensional homology groups* of X. We get the *augmented singular complex* of X

$$\cdots \longrightarrow S_2(X) \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\varepsilon} \mathbf{Z} \longrightarrow 0.$$

It will prove to be useful to consider the reduced homology groups only for $X \neq \emptyset$.

We will now examine the relation between $H_0(X)$ and $\tilde{H}_0(X)$. Remark that $\tilde{Z}_0(X)$ is a subgroup of $Z_0(X) = S_0(X)$ and that therefore $\tilde{H}_0(X)$ is a subgroup of $H_0(X)$. Denote the inclusion homomorphism by $\xi : \tilde{H}_0(X) \to H_0(X)$. Further, from $\varepsilon\partial_1 = 0$ we know that $\varepsilon(B_0(X)) = 0$. Hence ε induces a homomorphism $\varepsilon_* : H_0(X) \to \mathbf{Z}$. We see by an easy argument that the sequence

$$0 \longrightarrow \tilde{H}_0(X) \xrightarrow{\xi} H_0(X) \xrightarrow{\varepsilon_*} \mathbf{Z} \longrightarrow 0$$

is exact. From this fact we conclude that there is a decomposition of $H_0(X)$ as

$$H_0(X) \cong \tilde{H}_0(X) \oplus \mathbf{Z}.$$

As a first example for the computation of homology groups we examine the case where X consists of a single point. The result is given in the next theorem which is also known as the dimension axiom.

Theorem 1.7 *If X is a space consisting only of one point then $H_q(X) = 0$ for $q > 0$ and $H_0(X) \cong \mathbf{Z}$.*

Proof. Because for each q there is only one map $f : \Delta_q \to X$, namely the constant map, we have only one singular simplex σ_q for each dimension q. The boundary of σ_q has the form

$$\partial \sigma_q = \sum_{i=0}^{q} (-1)^i \sigma_q \delta^i_{q-1} = \sum_{i=0}^{q} (-1)^i \sigma_{q-1}$$

because $\sigma_q \delta^i_{q-1}$ is necessarily the only $(q-1)$-simplex σ_{q-1}. It follows that

$$\partial \sigma_q = \begin{cases} 0 & \text{if } q \text{ is odd or } q = 0 \\ \sigma_{q-1} & \text{if } q \text{ is even, } q > 0 \end{cases}$$

For q odd, $q > 0$ we get $B_q(X) \cong Z_q(X)$ and for q even, $q > 0$ we get $Z_q(X) = 0$. Therefore in both cases $H_q(X) = 0$. Finally $B_0(X) = 0$ and $Z_0(X)$ is generated by σ_0. Hence $H_0(X) \cong \mathbf{Z}$. $\square$

From Theorem 1.7 we get at once that for a one-point space X we have

$$\tilde{H}_q(X) = 0$$

for all q. A space X with this property is called *acyclic*.

Theorem 1.8 *Let X_γ, $\gamma \in \Gamma$, denote the set of path connected components of X. Then there is a canonical isomorphism*

$$H_q(X) \cong \bigoplus_\gamma H_q(X_\gamma)$$

for all q.

Proof. Each singular q-simplex lies entirely in one of the arc components. Therefore we have an isomorphism

$$S_q(X) \cong \bigoplus_\gamma S_q(X_\gamma)$$

for all q. The boundary operates component by component. Therefore we have the direct sum decompositions

$$Z_q(X) \cong \bigoplus_\gamma Z_q(X_\gamma) \text{ and } B_q(X) \cong \bigoplus_\gamma B_q(X_\gamma)$$

Forming the quotient $H_q(X) = Z_q(X)/B_q(X)$ gives the result. $\square$

Theorem 1.9 *Let X be a nonempty space. Then $H_0(X)$ is a free abelian group whose rank is equal to the number of path components of X.*

Proof. By Theorem 1.8, we may assume path connectedness of X. Observe that $\varepsilon : S_0(X) \to \mathbf{Z}$ is an epimorphism. We claim that $B_0(X) = \ker \varepsilon$. From this we directly get $H_0(X) \cong \mathbf{Z}$ as desired. Now we prove the claim. From $\varepsilon\partial_1 = 0$ we get $B_0(X) \subseteq \ker \varepsilon$. For $\ker \varepsilon \subseteq B_0(X)$ choose a basepoint x_0 in X. For $x \in X$ let σ_x be a path from x_0 to x such that $\partial \sigma_x = x - x_0$. Now, given a cycle $c = \sum_x n_x x \in \ker \varepsilon$ we get

$$c = \sum_x n_x x = \sum_x n_x x - \sum_x n_x x_0 = \partial(\sum_x n_x \sigma_x).$$

Hence $c \in B_0(X)$. □

1.3 The Homomorphism Induced by a Continous Map

In the following our aim is to show that for any continous map $f : X \to Y$ between topological spaces X and Y we can associate a sequence of homomorphisms $f_* : H_q(X) \to H_q(Y)$ for every q. We start to give the necessary definitions.

Definition 1.10 Let $f : X \to Y$ be a continous map and $\sigma : \Delta_q \to X$ a q-simplex in X. Then $f \circ \sigma : \Delta_q \to Y$ is a q-simplex in Y and we can define a homomorphism $f_\# : S_q(X) \to S_q(Y)$ by

$$f_\#(\sum_\sigma n_\sigma \sigma) = \sum_\sigma n_\sigma (f \circ \sigma).$$

The notation is a bit careless for there is a different $f_\#$ for every q.

Lemma 1.11 *The following diagram is commutative*

$$\begin{array}{ccc} S_q(X) & \xrightarrow{\partial_q} & S_{q-1}(X) \\ \downarrow f_\# & & \downarrow f_\# \\ S_q(Y) & \xrightarrow{\partial_q} & S_{q-1}(Y) \end{array}$$

for every q. We can also write this as $\partial f_\# = f_\# \partial$.

Proof. It suffices to show commutativity for every q-simplex σ in X. We get

$$\begin{aligned} f_\#\partial\sigma &= f_\#(\sum_{i=0}^{q}(-1)^i \sigma\partial^i_{q-1}) \\ &= \sum_{i=0}^{q}(-1)^i f_\#(\sigma\partial^i_{q-1}) = \sum_{i=0}^{q}(-1)^i f\sigma\partial^i_{q-1} \text{ and} \\ \partial f_\#\sigma &= \partial(f\sigma) = \sum_{i=0}^{q}(-1)^i f\sigma\partial^i_{q-1}. \end{aligned}$$

□

Lemma 1.12 *For every q we have*

$$\begin{aligned} f_{\#}(Z_q(X)) &\subseteq Z_q(Y) \text{ and} \\ f_{\#}(B_q(X)) &\subseteq B_q(Y). \end{aligned}$$

Proof. Let $z \in Z_q(X)$. Then $\partial z = 0$ and therefore $\partial f_{\#} z = f_{\#} \partial z = 0$, i.e. $f_{\#} z \in Z_q(Y)$. Let now $b \in B_q(X)$. Then $b = \partial c$ for some $c \in S_{q+1}(X)$. Hence $f_{\#} b = f_{\#} \partial c = \partial f_{\#} c \in B_q(Y)$. □

Because of the last lemma $f_{\#}$ induces a homomorphism of quotient groups, which we denote by

$$f_* : H_q(X) \to H_q(Y).$$

We see at once that $\varepsilon = f_{\#}\varepsilon$ and hence $f_{\#}(\tilde{Z}_0(X)) \subseteq \tilde{Z}_0(Y)$ and we get a homomorphism

$$f_* : \tilde{H}_0(X) \to \tilde{H}_0(Y).$$

It is easily checked that for continous maps $f, g : X \to Y$ we get $(fg)_* = f_* g_*$.

2 The Exact Homology Sequence of a Pair

To be able to use homology efficiently we need some tools to actually determine the homology groups of various spaces. In this section we investigate how the homology of X does depend on the homology of a subspace A of X. We define the concept of relative homology groups which is a generalization of the earlier defined "absolute" homology groups and get an answer to the mentioned question in form of an exact sequence of the pair (X, A).

We now make the necessary definitions. Let A be a subspace of the topological space X. We can consider the chains in A which are also chains in X. Therefore $S_q(A)$ can be regarded as a subgroup of $S_q(X)$ and we can form the quotient $S_q(X, A) = S_q(X)/S_q(A)$, which we will call the group of *relative singular q-chains*. Because ∂_q has the property that $\partial_q(S_q(A)) \subseteq S_{q-1}(A)$ it induces a *boundary operator* (also denoted by ∂_q) on the quotient group

$$\partial_q : S_q(X, A) \to S_{q-1}(X, A).$$

We define the group of *relative q-cycles* as $Z_q(X, A) = \ker \partial_q$ and the group of *relative q-boundaries* as $B_q(X, A) = \operatorname{im} \partial_{q+1}$. We have $B_q(X, A) \subseteq Z_q(X, A)$ and can therefore form the quotient

$$H_q(X, A) = Z_q(X, A)/B_q(X, A)$$

which we call the q-th relative homology group. Remark that relative homology groups are a generalization of the earlier defined homology groups of a space X. If we let $A = \emptyset$ we get $H_q(X) = H_q(X, A)$.

Corresponding to the inclusion map $i : A \to X$ we have the induced inclusion homomorphism

$$i_* : H_q(A) \to H_q(X).$$

Similarly, by regarding each q-cycle as a relative q-cycle we get a homomorphism

$$j_* : H_q(X) \to H_q(X, A).$$

Lastly we define the boundary operator $\partial_* : H_q(X, A) \to H_{q-1}(A)$ of the pair (X, A) to be the following homomorphism

$$\partial_*(\{z\}_{(X,A)}) = \{\partial z\}_A.$$

This last definition requires justification. If $z \in Z_q(X, A)$ then $\partial z \in S_{q-1}(A)$. But of course ∂z is a cycle, so $\{\partial z\}_A \in H_{q-1}(A)$ is defined. To see that the definition is independent of the choice of the representative cycle z let z' be a cycle homologous to z relative A. Then there is a $c \in S_{q+1}(X)$ and a $c' \in S_q(A)$ such that $z' - z = \partial c + c'$. Therefore $\partial z' = \partial z + \partial c'$ and hence $\{\partial z'\}_A = \{\partial z\}_A$.

Using the homomorphisms i_*, j_* and ∂_* we can construct the following sequence

$$\cdots \xrightarrow{j_*} H_{q+1}(X, A) \xrightarrow{\partial_*} H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial_*} \cdots.$$

We call the sequence the *homology sequence of the pair* (X, A).

Theorem 2.1 *The homology sequence of the pair (X, A) is exact.*

Proof. The proof consists of three parts.

1. $\ker i_* = \operatorname{im} \partial_*$
2. $\ker j_* = \operatorname{im} i_*$
3. $\ker \partial_* = \operatorname{im} j_*$

1. Let $c_q \in Z_q(A)$. Then c_q represents an element in $\ker i_*$ precisely if $i_*(c_q)$ is a boundary in $S_q(X)$, i.e. if there is a $c_{q+1} \in S_{q+1}(X)$ such that $\partial c_{q+1} = i_*(c_q)$, i.e. $\partial c_{q+1} = c_q$ if c_q is regarded as a chain in $S_q(X)$. But the $(q+1)$-chains c_{q+1} of $S_{q+1}(X)$ with the property that ∂c_{q+1} is a cycle in $S_q(A)$ are precisely those representing q-cycles of $S_{q+1}(X, A)$. Therefore $\ker i_* = \operatorname{im} \partial_*$.
2. Let $c_q \in Z_q(X)$ such that $j_*(\{c_q\}) = 0$. Then there is a $c_{q+1} \in S_{q+1}(X)$ and a $c'_q \in S_q(A)$ such that $c_q = c'_q + \partial c_{q+1}$. Hence a cycle c_q is in $\ker j_*$ precisely if there is a $c'_q \in S_q(A)$ which is homologous to it. But this means that c_q represents an element of $\operatorname{im} i_*$.

3. Let $\{c_q\} \in H_q(X,A)$ such that $\partial_*\{c_q\} = 0$. This is equivalent to the existence of a $c'_q \in Z_q(X)$ and a $c''_q \in S_q(A)$ such that $c_q = c'_q + c''_q$. But this means the existence of a cycle $c'_q = c_q - c''_q \in Z_q(X)$ which represents the same element of $H_q(X,A)$ as c_q does, i.e. $\{c'_q\}_{(X,A)} = \{c_q\}_{(X,A)}$. This implies that $\{c_q\} \in \operatorname{im} j_*$.

$\square$

The sequence of homomorphisms remains exact if we replace $H_q(X)$ by $\tilde{H}_q(X)$ and $H_q(A)$ by $\tilde{H}_q(A)$. We also state without proof another theorem which expresses the relations between reduced homology and relative homology groups.

Theorem 2.2 *Let $x_0 \in X$. Then*

$$\tilde{H}_q(X) \cong H_q(X, x_0)$$

for all q.

For later use we also need the homology sequence of a triad. We give the result without proof.

Theorem 2.3 *Let $B \subset A \subset X$ be subspaces of X and let $i : (A,B) \to (X,B)$ and $j : (X,B) \to (X,A)$ be inclusions. Then the following sequence, called the sequence of the triad (X,A,B),*

$$\cdots \xrightarrow{j_*} H_{q+1}(X,A) \xrightarrow{\partial_*} H_q(A,B) \xrightarrow{i_*} H_q(X,B) \xrightarrow{j_*} H_q(X,A) \xrightarrow{\partial_*} \cdots$$

is exact. The boundary operator ∂_* is given by $\partial_*(\{z\}_{(X,A)}) = \{\partial z\}_{(A,B)}$.

3 The Excision Property

We now come to a very important but quite subtle property of relative homology groups. Intuitivly speaking when forming the quotient $S_q(X,A) = S_q(X)/S_q(A)$ we forget about everything inside A. We could therefore hope that $H_q(X,A)$ only depends on $X \setminus A$. The actual statement is a bit weaker. The proof which involves barycentric subdivision is quite lengthy and will be omitted.

Theorem 3.1 *Let $U \subset A \subset X$ be subspaces with $\overline{U} \subset A^o$. Then the inclusion $i : (X \setminus U, A \setminus U) \to (X,A)$ induces isomorphisms*

$$i_* : H_q(X \setminus U, A \setminus U) \to H_q(X,A)$$

for all q.

We can also formulate the excision property in a different way, in which it is used quite often.

Theorem 3.2 *Let X_1 and X_2 be subspaces of X such that $X = X_1^\circ \cup X_2^\circ$. Then the inclusion $j : (X_1, X_1 \cap X_2) \to (X, X_2)$ induces isomorphisms*

$$j_* : H_q(X_1, X_1 \cap X_2) \to H_q(X, X_2)$$

for all q.

Proof. We use Theorem 3.1. Let $U = X \setminus X_1$ and $A = X_2$. First, we show that $\overline{U} \subset A^\circ$. From $X_1^\circ \subset X_1$ we have $X \setminus X_1 \subset X \setminus X_1^\circ$. Therefore $\overline{U} = \overline{(X \setminus X_1)} \subset X \setminus X_1^\circ$, because $X \setminus X_1^\circ$ is closed. Further

$$X \setminus X_1^\circ = (X_1^\circ \cup X_2^\circ) \setminus X_1^\circ = X_2^\circ \setminus X_1^\circ \subset X_2^\circ = A^\circ.$$

Second, we prove that $(X \setminus U, A \setminus U)$ is the same as $(X_1, X_1 \cap X_2)$. We have

$$\begin{aligned} X \setminus U &= X \setminus (X \setminus X_1) = X_1 \text{ and} \\ A \setminus U &= X_2 \setminus (X \setminus X_1) = X_1 \cap X_2. \end{aligned}$$

Obviously (X, A) is the same as (X, X_2) and therefore $i = j$ which gives $i_* = j_*$. □

4 The Homotopy Theorem

We will see in the following that homology groups possess another very important property, namely their invariance for a large class of spaces. The homotopy theorem (also called the homotopy axiom) states that spaces from the same homotopy class have isomorphic homology groups. This also simplifies the computation of homology groups. To determine $H_q(X)$ we replace X by a simpler space $Y \simeq X$ and compute $H_q(Y) \cong H_q(X)$. The main result is the following:

Theorem 4.1 *Let $f, g : X \to Y$ be continous maps. If f and g are homotopic then the induced homomorphisms f_* and g_* of $H_q(X)$ into $H_q(Y)$ are the same.*

The theorem also holds for reduced homology groups, i.e. $f_* = g_* : \tilde{H}_q(X) \to \tilde{H}_q(Y)$. We will not give the proof here, it can be found in any of the text books about homology. Instead we continue with some corollaries.

Theorem 4.2 *If $f : X \to Y$ is a homotopy equivalence then $f_* : H_q(X) \to H_q(Y)$ are isomorphisms for all q. We also have isomorphisms $f_* : \tilde{H}_q(X) \to \tilde{H}_q(Y)$.*

Proof. We have a $g : Y \to X$ with $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$. Using Theorem 4.1 we get $(gf)_* = g_* f_* = \mathrm{id}$ and $(fg)_* = f_* g_* = \mathrm{id}$. Hence $f_*^{-1} = g_*$ and f_* is bijective. □

Using the exact sequence of the pair (X, A) we obtain from Theorem 4.2:

Corollary 4.3 *If $A \subset X$ is a deformation retract of X then $i_* : H_q(A) \to H_q(X)$ is an isomorphism for all q. Also $H_q(X, A) = 0$.*

By considering the exact sequence of the triad $B \subset A \subset X$ and application of the last corollary and Theorem 4.2 we conclude:

Corollary 4.4 *If $B \subset A \subset X$ and B is a deformation retract of A then we have isomorphisms $j_* : H_q(X, B) \to H_q(X, A)$.*

The homotopy axiom enables us to determine the homology of contractible spaces:

Corollary 4.5 *If X is a contractible space then $\tilde{H}_q(X) = 0$ for all $q \geq 0$.*

Proof. X has the same homotopy type as a one-point space. Application of Theorem 4.2 and the dimension axiom (Theorem 1.7) gives the result. $\square$

From this we can conclude that $\mathbf{R}^n$ is acyclic.

5 The Mayer-Vietoris Sequence

In this section we discuss a very powerful tool for determining the homology groups of many spaces: the Mayer-Vietoris sequence. Suppose that a space $X = X_1 \cup X_2$ is given as the union of two subspaces. How does the homology of X depend on X_1 and X_2? The answer will be given in form of an exact sequence, the Mayer-Vietoris sequence, which plays the same role for homology groups as the Seifert-Van Kampen theorem does for the fundamental group.

Before we come to the main result we will prove a lemma.

Lemma 5.1 (Barratt-Whitehead) *Given a diagram with exact rows in which all rectangles commute*

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow & A_q & \xrightarrow{f_q} & B_q & \xrightarrow{g_q} & C_q & \xrightarrow{h_q} & A_{q-1} & \xrightarrow{f_{q-1}} & B_{q-1} & \longrightarrow \cdots \\
 & \downarrow \alpha_q & & \downarrow \beta_q & & \downarrow \gamma_q & & \downarrow \alpha_{q-1} & & \downarrow \beta_{q-1} & \\
\cdots \longrightarrow & A'_q & \xrightarrow{f'_q} & B'_q & \xrightarrow{g'_q} & C'_q & \xrightarrow{h'_q} & A'_{q-1} & \xrightarrow{f'_{q-1}} & B'_{q-1} & \longrightarrow \cdots
\end{array}$$

if all the γ_q are isomorphisms then there is an exact sequence

$$\cdots \xrightarrow{\Gamma_{q+1}} A_q \xrightarrow{\Phi_q} B_q \oplus A'_q \xrightarrow{\Psi_q} B'_q \xrightarrow{\Gamma_q} A_{q-1} \xrightarrow{\Phi_{q-1}} \cdots$$

where $\Phi_q(a) = (f_q(a), \alpha_q(a))$, $\Psi_q(b, a') = \beta_q(b) - f'_q(a')$ and $\Gamma_q(b') = h_q \gamma_q^{-1} g'_q$. This latter sequence is called the Barrett-Whitehead sequence of the ladder.

Proof. The proof of exactness is a diagram chase. We will first prove exactness at B'_q. We have to show that $\operatorname{im}\Psi_q = \ker\Gamma_q$. For $\operatorname{im}\Psi_q \subseteq \ker\Gamma_q$ we need $\Gamma_q\Psi_q(b,a') = 0$. But

$$\Gamma_q\Psi_q(b,a') = \Gamma_q(\beta_q(b) - f'_q(a')) = h_q\gamma_q^{-1}g'_q\beta_q(b) - h_q\gamma_q^{-1}g'_qf'_q(a').$$

The first term is 0 because $h_q\gamma_q^{-1}g'_q\beta_q = g_qh_q = 0$ and the second term is 0 because of $g'_qf'_q = 0$.

For $\ker\Gamma_q \subseteq \operatorname{im}\Psi_q$ take $b' \in B'_q$ such that $\Gamma_q(b') = 0$. Because of the exactness of the upper row and $\gamma_q^{-1}g'_q(b) \in \ker h_q$ there exists $b \in B_q$ such that $g_q(b) = \gamma_q^{-1}g'_q(b')$. By commutativity we get $g'_q(b' - \beta_q b) = 0$. Therefore, by exactness of the lower row there is an $a' \in A'_q$ such that $f'_q(a') = b' - \beta_q b$. Then we have

$$\Psi_q(b, -a') = \beta_q(b) + f'_q(a') = b'$$

as desired. In a similar manner we get $\operatorname{im}\Phi_q = \ker\Psi_q$ and $\operatorname{im}\Gamma_q = \ker\Phi_{q-1}$. $\square$

We are now able to prove the theorem about the Mayer-Vietoris sequence.

Theorem 5.2 (Mayer-Vietoris) *Let X_1 and X_2 be subspaces of the topological space X such that $X = X_1^\circ \cup X_2^\circ$. Then there is an exact sequence, called the Mayer-Vietoris sequence of X*

$$\cdots \xrightarrow{\Delta} H_q(X_1\cap X_2) \xrightarrow{\phi} H_q(X_1)\oplus H_q(X_2) \xrightarrow{\psi} H_q(X) \xrightarrow{\Delta} H_{q-1}(X_1\cap X_2) \xrightarrow{\phi} \cdots$$

Here

$$\begin{array}{ll} \phi(x) = (i_*(x), j_*(x)) & x \in H_q(X_1\cap X_2) \\ \psi(x_1,x_2) = k_*(x_1) - l_*(x_2) & x_1 \in H_q(X_1), x_2 \in H_q(X_2) \end{array}$$

with the homomorphisms i_*, j_*, k_* and l_* induced by the inclusions

$$i : X_1 \cap X_2 \to X_1,\ j : X_1 \cap X_2 \to X_2,\ k : X_1 \to X \text{ and } l : X_2 \to X.$$

Finally $\Delta = dh_*^{-1}q_*$ with h and q inclusions and d the boundary operator of the pair $(X_1, X_1 \cap X_2)$.

Proof. Consider the following diagram of pairs of spaces where all maps are inclusions

$$\begin{array}{ccccc} (X_1\cap X_2, \emptyset) & \xrightarrow{i} & (X_1,\emptyset) & \xrightarrow{p} & (X_1, X_1\cap X_2) \\ \downarrow j & & \downarrow k & & \downarrow h \\ (X_2,\emptyset) & \xrightarrow{l} & (X,\emptyset) & \xrightarrow{q} & (X, X_2) \end{array}$$

This diagram is commutative. From this we get another diagram where the rows are exact by Theorem 2.1:

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & H_q(X_1\cap X_2) & \xrightarrow{i_*} & H_q(X_1) & \xrightarrow{p_*} & H_q(X_1, X_1\cap X_2) & \xrightarrow{d} & H_{q-1}(X_1\cap X_2) & \longrightarrow \cdots \\ & \downarrow j_* & & \downarrow k_* & & \downarrow h_* & & \downarrow j_* & \\ \cdots \longrightarrow & H_q(X_2) & \xrightarrow{l_*} & H_q(X) & \xrightarrow{q_*} & H_q(X, X_2) & \longrightarrow & H_{q-1}(X_2) & \longrightarrow \cdots \end{array}$$

From the excision property (Theorem 3.2) we conclude that h_* is an isomorphism. Application of Lemma 5.1 gives the result. $\square$

We can also formulate Theorem 5.2 for reduced homology groups an obtain

Theorem 5.3 *Let X_1 and X_2 be subspaces of the topological space X such that $X = X_1^\circ \cup X_2^\circ$. Then there is an exact sequence*

$$\cdots \xrightarrow{\Delta} \tilde{H}_q(X_1 \cap X_2) \xrightarrow{\phi} \tilde{H}_q(X_1) \oplus \tilde{H}_q(X_2) \xrightarrow{\psi} \tilde{H}_q(X) \xrightarrow{\Delta} \tilde{H}_{q-1}(X_1 \cap X_2) \xrightarrow{\phi} \cdots$$

The sequence ends with

$$\cdots \xrightarrow{\phi} \tilde{H}_0(X_1) \oplus \tilde{H}_0(X_2) \xrightarrow{\psi} \tilde{H}_0(X) \xrightarrow{\Delta} 0.$$

Proof. Let $x_0 \in X$. Consider the commutative diagram with all maps inclusions

$$\begin{array}{ccccc} (X_1 \cap X_2, x_0) & \xrightarrow{i} & (X_1, x_0) & \xrightarrow{p} & (X_1, X_1 \cap X_2) \\ \downarrow j & & \downarrow k & & \downarrow h \\ (X_2, x_0) & \xrightarrow{l} & (X, x_0) & \xrightarrow{q} & (X, X_2) \end{array}$$

Using Theorem 2.2 then proof then proceeds as in Theorem 5.2. $\square$

6 Examples

After having examined some techniques of homology theory we will now use these methods to determine the homology of a few spaces. We know already that $\tilde{H}_q(\mathbf{R}^n) = 0$ for all $q \geq 0$. Next we turn to S^n.

Theorem 6.1 *For the sphere S^n, $n \geq 0$ we have*

$$\tilde{H}_q(S^n) \cong \begin{cases} \mathbf{Z} & \text{if } q = n \\ 0 & \text{if } q \neq n \end{cases}$$

Proof. The proof is by induction on n. S^0 consists of two points and by Theorem 1.7 (dimension axiom) and Theorem 1.8 we get $H_0(S^0) \cong \mathbf{Z}^2$ and hence $\tilde{H}_0(S^0) \cong \mathbf{Z}$ and $\tilde{H}_q(S^0) = 0$ for $q > 0$.

Assume now $n > 0$. Let P_+ be the north pole and P_- be the south pole of S^n and let $X_1 = S^n \setminus P_+$ and $X_2 = S^n \setminus P_-$. Observe that $X_1^\circ \cup X_2^\circ = S^n$. Consider the corresponding Mayer-Vietoris sequence of S^n:

$$\tilde{H}_q(X_1) \oplus \tilde{H}_q(X_2) \longrightarrow \tilde{H}_q(S^n) \longrightarrow \tilde{H}_{q-1}(X_1 \cap X_2) \longrightarrow \tilde{H}_{q-1}(X_1) \oplus \tilde{H}_{q-1}(X_2).$$

Here X_1 and X_2 are contractible and S^{n-1} is a deformation retract of $X_1 \cap X_2$. Therefore

$$0 \longrightarrow \tilde{H}_q(S^n) \longrightarrow \tilde{H}_{q-1}(S^{n-1}) \longrightarrow 0$$

is exact. This means that $\tilde{H}_q(S^n) \cong \tilde{H}_{q-1}(S^{n-1})$ and the proof is complete. $\square$

An important application of this is the following corollary known as the invariance of dimension.

Corollary 6.2 *If $n \neq m$, then $\mathbf{R}^n$ and $\mathbf{R}^m$ are not homeomorphic.*

Proof. If there was a homeomorphism between $\mathbf{R}^n$ and $\mathbf{R}^m$ there would also be a homeomorphism between the one-point compactifications of $\mathbf{R}^n$ and $\mathbf{R}^m$, namely between S^n and S^m. But $H_n(S^n) \neq H_n(S^m)$ for $n \neq m$. $\square$

In order to illustrate how homology groups can be used to distinguish efficiently between different topological spaces we will just give some more examples without proof.

Torus T	$H_0(T) \cong \mathbf{Z}$
	$H_1(T) \cong \mathbf{Z^2}$
	$H_2(T) \cong \mathbf{Z}$
Torus with a Hole H	$H_0(H) \cong \mathbf{Z}$
	$H_1(H) \cong \mathbf{Z^2}$
	$H_2(H) = 0$
Klein Bottle K	$H_0(K) \cong \mathbf{Z}$
	$H_1(K) \cong \mathbf{Z} \oplus \mathbf{Z_2}$
	$H_2(K) = 0$
Projective Plane P	$H_0(P) \cong \mathbf{Z}$
	$H_1(P) \cong \mathbf{Z_2}$
	$H_2(P) = 0$
Möbius Strip M	$H_0(M) \cong \mathbf{Z}$
	$H_1(M) \cong \mathbf{Z}$
	$H_2(M) = 0$

In particular it should be observed that for the torus the first homology group is a free abelian group and the second is infinite cyclic whereas for the Klein bottle the first group contains a cyclic subgroup of order 2 and the second homology group is 0. This behaviour is quite typical and can be used to distinguish between closed surfaces which are orientable as the torus and those which are nonorientable as the Klein bottle or the projective plane.

Part II
Separation Theorems

7 The Jordan-Brouwer Separation Theorem

In order to prove the Jordan-Brouwer separation theorem we need the following lemma, which is of fundamental importance for this chapter.

Lemma 7.1 *Let $B \subset S^n$ be a subset of S^n which is homeomorphic to I^k where $0 \leq k \leq n$. Then $\tilde{H}_q(S^n \setminus B) = 0$ for all q.*

Proof. The proof is by induction on k. For $k = 0$ the set B is a single point and $S^n \setminus B \approx \mathbf{R}^n$, which is acyclic. Suppose now that the theorem holds for $k - 1$. Let

$$z \in \tilde{Z}_q(S^n \setminus B).$$

We want to prove that $z = \partial b$ for some $b \in S_{q+1}(S^n \setminus B)$. This would imply $B_q(S^n \setminus B) \cong \tilde{Z}_q(S^n \setminus B)$ and hence $\tilde{H}_q(S^n \setminus B) = 0$ as desired.

We assume that we have chosen a fixed homeomorphism $f : I^{k-1} \times I \to B$. Let

$$B_t = f(I^{k-1} \times t) \subset B \subset S^n.$$

Then B_t is a $(k-1)$-ball and hence $\tilde{H}_q(S^n \setminus B_t) = 0$ by the inductive hypothesis.

Clearly $z \in \tilde{Z}_q(S^n \setminus B_t)$ and by the hypothesis we have a $b_t \in S_{q+1}(S^n \setminus B_t)$ with $\partial b_t = z$. We know that b_t is of the form $b_t = n_1\sigma_1 + \ldots + n_l\sigma_l$ where $\sigma_i : \Delta_{q+1} \to S^n \setminus B_t$. Note that $L = \bigcup_{i=1}^{l} \sigma_i(\Delta_{q+1})$ is compact and $L \cap B_t = \emptyset$. Therefore we have an open neighbourhood U_t of B_t with $L \cap U_t = \emptyset$. Observe that $b_t \in S_{q+1}(S^n \setminus U_t)$. Since $I^{k-1} \times t \subset f^{-1}(B \cap U_t)$ we also have a neighbourhood V_t of t with $I^{k-1} \times V_t \subset f^{-1}(B \cap U_t)$. We choose m big enough such that for every closed interval $I_j = [\frac{j-1}{m}, \frac{j}{m}]$ there is a t_j with $I_j \subset V_{t_j}$. Let $Q_j = f(I^{k-1} \times I_j)$. We have $Q_j \subset U_{t_j}$ and $B = \bigcup_{j=1}^{m} Q_j$ and for every j there is a $b_{t_j} \in S_{q+1}(S^n \setminus Q_j)$ with $t = \partial b_{t_j}$.

Let $X_1 = S^n \setminus Q_1$ and $X_2 = S^n \setminus Q_2$. Then

$$\begin{aligned} X_1 \cup X_2 &= S^n \setminus (Q_1 \cap Q_2) = S^n \setminus B_{1/m} \text{ and} \\ X_1 \cap X_2 &= S^n \setminus (Q_1 \cup Q_2) \end{aligned}$$

$B_{1/m}$ is a $(k-1)$-ball and Q_1, Q_2 and $Q_1 \cup Q_2$ are k-balls. Consider now the exact Mayer-Vietoris sequence of $S^n \setminus B_{1/m}$:

$$\tilde{H}_{q+1}(X_1 \cup X_2) \xrightarrow{\Delta} \tilde{H}_q(X_1 \cap X_2) \xrightarrow{\phi} \tilde{H}_q(X_1) \oplus \tilde{H}_q(X_2) \xrightarrow{\psi} \tilde{H}_q(X_1 \cup X_2)$$

But $\tilde{H}_{q+1}(X_1 \cup X_2) \cong \tilde{H}_q(X_1 \cup X_2) = 0$ and hence ϕ is an isomorphism. For $z \in \tilde{Z}_q(X_1 \cap X_2)$ we get $\phi(\{z\}) = 0$ because $z = \partial b_{t_1}$ in X_1 and $z = \partial b_{t_2}$

in X_2 and therefore also $\{z\} = 0$. But this means that $z = \partial b$ for some $b \in S_{q+1}(S^n \setminus (Q_1 \cup Q_2))$.

The same argument as above is applied to $X_1 = S^n \setminus (Q_1 \cup Q_2)$ and $X_2 = S^n \setminus Q_3$. Examination of the Mayer-Vietoris sequence gives that z is the boundary of some chain in $S^n \setminus (Q_1 \cup Q_2 \cup Q_3)$. By iterating this procedure we finally get that $z = \partial b$ for some $b \in S_{q+1}(S^n \setminus (Q_1 \cup \ldots \cup Q_m)) = S_{q+1}(S^n \setminus B)$, which completes the proof. $\square$

In order to illustrate why this lemma is so important and possesses such a somewhat complicated proof we can consider some examples of wild arcs $B \subset S^3$ where $B \approx I^1$ and $S^3 \setminus B$ has a nontrivial fundamental group. For the definition of wild see Section 8.

Theorem 7.2 *Let $S \subset S^n$ be a subset of S^n which is homeomorphic to S^k where $0 \leq k \leq n-1$. Then $\tilde{H}_{n-k-1}(S^n \setminus S) \cong \mathbf{Z}$ and $\tilde{H}_q(S^n \setminus S) = 0$ for all $q \neq n-k-1$.*

Proof. The proof is by induction on k. For $k = 0$ the subset S consists of two points and hence $S^n \setminus S$ is homeomorphic to $\mathbf{R}^n$ with one point removed which is homotopic to S^{n-1} and so $\tilde{H}_{n-1}(S^n \setminus S) \cong \mathbf{Z}$ and $\tilde{H}_q(S^n \setminus S) = 0$ for $q \neq n-1$.

Assume now that the theorem holds for $k-1$. We can fix a homeomorphism $f : S^k \to S$. Let D^k_+ and D^k_- denote the upper and the lower hemisphere of S^k,respectively. Then $B_1 = f(D^k_+)$ and $B_2 = f(D^k_-)$ are k-balls and $S = B_1 \cup B_2$. Let $T = B_1 \cap B_2$. We know that T is of the same homotopy class as S^{k-1}. Remark that

$$\begin{aligned}(S^n \setminus B_1) \cup (S^n \setminus B_2) &= S^n \setminus T \text{ and}\\ (S^n \setminus B_1) \cap (S^n \setminus B_2) &= S^n \setminus S.\end{aligned}$$

Application of the Mayer-Vietoris sequence gives

$$\begin{aligned}\tilde{H}_{q+1}(S^n \setminus B_1) \oplus \tilde{H}_{q+1}(S^n \setminus B_2) &\longrightarrow \tilde{H}_{q+1}(S^n \setminus T) \longrightarrow \tilde{H}_q(S^n \setminus S)\\ &\longrightarrow \tilde{H}_q(S^n \setminus B_1) \oplus \tilde{H}_q(S^n \setminus B_2).\end{aligned}$$

But from Lemma 7.1 we know that $\tilde{H}_q(S^n \setminus B_1) \cong \tilde{H}_q(S^n \setminus B_2) = 0$ for all $q \geq 0$ and therefore $\tilde{H}_{q+1}(S^n \setminus T) \cong \tilde{H}_q(S^n \setminus S)$ which proves the inductive step. $\square$

From this we can directly conclude

Theorem 7.3 (Jordan-Brouwer Separation Theorem) *Let $S \subset S^n$ be a subset which is homeomorphic to S^{n-1}. Then $S^n \setminus S$ has exactly two components.*

For $n = 2$ this is known as the Jordan curve theorem. For $n > 2$ we have the Brouwer separation theorem.

Proof. Applying Theorem 7.2 for $k = n-1$ gives $\tilde{H}_0(S^n \setminus S) \cong \mathbf{Z}$ and hence $H_0(S^n \setminus S) \cong \mathbf{Z}^2$ which means that $S^n \setminus S$ has two arc components. But because $S^n \setminus S$ is locally arcwise connected the arc components are the same as the components. $\square$

We can also conclude from Theorem 7.2 that a homeomorph of S^k where $k < n-1$ does not seperate S^n. The next statement is a stronger version of the Jordan-Brouwer separation theorem:

Theorem 7.4 *Let $S \subset S^n$ be a subset of S^n which is homeomorphic to S^{n-1}. Then S is the boundary of the two components U and V of $S^n \setminus S$.*

Proof. $S^n \setminus S$ is locally path connected and therefore U and V are open subsets of $S^n \setminus S$ and hence also of S^n. This shows that $\dot{U} \subset S$ and $\dot{V} \subset S$. For the reverse inclusion we must show that for every $x \in S$ we get $x \in \dot{U}$ and $x \in \dot{V}$.

Let N be an open neighbourhood of x. We have to prove that $N \cap U \neq \emptyset$ and $N \cap V \neq \emptyset$. $N \cap S$ is an open neighbourhood of x in S. We can find a decomposition of S as $S = B_1 \cup B_2$ where B_1 and B_2 are $(n-1)$-balls and $B_1 \cap B_2 \approx S^{n-2}$ such that $B_1 \subset N \cap S$. From Lemma 7.1 we conclude that $S^n \setminus B_2$ is path connected and hence we can choose a path in $S^n \setminus B_2$ from a point $p_1 \in U$ to $p_2 \in V$. Let $f : I \to S^n \setminus B_2$ be a continous map such that $f(0) = p_1$ and $f(1) = p_2$. Necessarily $f(I) \cap S \neq \emptyset$ and hence $f(I) \cap B_1 \neq \emptyset$. Let

$$t_0 = \inf\{t \in I : f(t) \in B_1\}.$$

Thus $f(t_0) \in f(I) \cap B_1 \subset N$. Consider now $J = [0, t_0)$. The set $f(J)$ is connected and contains $p_1 = f(0)$ and

$$f(J) \subset f(I) \cap (S^n \setminus S) = f(I) \cap (U \cup V).$$

Therefore $f(J) \subset U$. Hence any open neighbourhood of $f(t_0)$ in N meets U and so $N \cap U \neq \emptyset$. Similarly we get $N \cap V \neq \emptyset$ via considering $t_1 = \sup\{t \in I : f(t) \in B_1\}$. $\square$

It should be noted how Lemma 7.1 also contributed to this proof. To appreciate the theorem consider a subset S of S^n homeomorphic to $S^{n-1} \times I$. In this case we have two path connected components U and V. Here $\dot{U} \subset S$ and $\dot{V} \subset S$ but neither $\dot{U} = S$ nor $\dot{V} = S$.

In the following we examine what happens if we replace S^n by $\mathbf{R}^n$. The next lemma is the equivalent of Lemma 7.1.

Lemma 7.5 *Let $B \subset \mathbf{R}^n$ be a subset of $\mathbf{R}^n$, $n \geq 2$ which is homeomorphic to I^k where $0 \leq k \leq n$. Then $\tilde{H}_{n-1}(\mathbf{R}^n \setminus B) \cong \mathbf{Z}$ and $\tilde{H}_q(\mathbf{R}^n \setminus B) = 0$ for $q \neq n-1$.*

Proof. We have a homeomorphism f between $\mathbf{R}^n$ and $S^n \setminus P_+$ via stereographic projection (P_+ is the north pole). Let $A = f(B)$. The set $A \subset S^n$ is a k-ball and $P_+ \notin A$. Consider the sequence of the pair $(S^n \setminus A, S^n \setminus (A \cup P_+))$:

$$\tilde{H}_{q+1}(S^n \setminus A) \longrightarrow \tilde{H}_{q+1}(S^n \setminus A, S^n \setminus (A \cup P_+)) \longrightarrow \tilde{H}_q(S^n \setminus (A \cup P_+)) \longrightarrow \tilde{H}_q(S^n \setminus A).$$

From Lemma 7.1 we know that $S^n \setminus A$ is acyclic and hence

$$\tilde{H}_{q+1}(S^n \setminus A, S^n \setminus (A \cup P_+)) \cong \tilde{H}_q(S^n \setminus (A \cup P_+)).$$

By the excision property we get $\tilde{H}_{q+1}(S^n \setminus A, S^n \setminus (A \cup P_+)) \cong \tilde{H}_q(S^n, S^n \setminus P_+)$ and furthermore $\tilde{H}_{q+1}(S^n, S^n \setminus P_+) \cong \tilde{H}_{q+1}(S^n, P_-)$ by Corollary 4.4 since P_- is a deformation retract of $S^n \setminus P_+$. But $\tilde{H}_{q+1}(S^n, P_-)$ is easily determined from the sequence of (S^n, P_-) and we get

$$\tilde{H}_{q+1}(S^n, P_-) \cong \tilde{H}_{q+1}(S^n) \cong \begin{cases} \mathbf{Z} & \text{for } q = n-1 \\ 0 & \text{for } q \neq n-1 \end{cases}$$

Finally $\tilde{H}_q(S^n \setminus (A \cup P_+)) \cong \tilde{H}_q(\mathbf{R}^n \setminus B)$ because the spaces are homeomorphic. This leads to the desired conclusion. □

From Theorem 7.2 we now conclude an equivalent statement for $\mathbf{R}^n \setminus S$ instead of $S^n \setminus S$:

Theorem 7.6 *Let $S \subset \mathbf{R}^n$ be homeomorphic to S^k with $n \geq 2$ and $0 \leq k \leq n-1$. Then*

$$\tilde{H}_q(\mathbf{R}^n \setminus S) \cong \begin{cases} \mathbf{Z} & \text{for } q = n-1 \text{ and } q = n-k-1 \\ 0 & \text{for } q \neq n-1,\ n-k-1 \end{cases}$$

Proof. We proceed as in Lemma 7.5. By stereographic projection we get a homeomorphism between $\mathbf{R}^n$ and $S^n \setminus P_+$. Let S be mapped homeomorphically into A. Hence $\tilde{H}_q(S^n \setminus (A \cup P_+)) \cong \tilde{H}_q(\mathbf{R}^n \setminus B)$. As in the proof of Lemma 7.5 we consider the following sequence:

$$\tilde{H}_q(\mathbf{R}^n \setminus B) \cong \tilde{H}_q(S^n \setminus (A \cup P_+)) \xleftarrow{\partial_*} \tilde{H}_{q+1}(S^n \setminus A, S^n \setminus (A \cup P_+)) \overset{(1)}{\cong}$$
$$\tilde{H}_{q+1}(S^n, S^n \setminus P_+) \overset{(2)}{\cong} \tilde{H}_{q+1}(S^n, P_-) \overset{(3)}{\cong} \tilde{H}_{q+1}(S^n).$$

We get the first isomorphism (1) from the excision property, (2) from the fact that P_- is a deformation retract of $S^n \setminus P_+$ and (3) from the sequence of (S^n, P_-). Now for ∂_* consider the following part of the sequence of the pair $(S^n \setminus A, S^n \setminus (A \cup P_+))$ for $q \neq n-k-2,\ n-k-1$:

$$\tilde{H}_{q+1}(S^n \setminus A) \longrightarrow \tilde{H}_{q+1}(S^n \setminus A, S^n \setminus (A \cup P_+)) \xrightarrow{\partial_*} \tilde{H}_q(S^n \setminus (A \cup P_+)) \longrightarrow \tilde{H}_q(S^n \setminus A).$$

But from Theorem 7.2 we know that $\tilde{H}_{q+1}(S^n \setminus A) \cong \tilde{H}_q(S^n \setminus A) = 0$ and hence ∂_* is an isomorphism in this case. The remaining part is

$$\tilde{H}_{n-k}(S^n \setminus A, S^n \setminus (A \cup P_+)) \xrightarrow{\partial_*} \tilde{H}_{n-k-1}(S^n \setminus (A \cup P_+)) \longrightarrow \tilde{H}_{n-k-1}(S^n \setminus A) \longrightarrow$$
$$\tilde{H}_{n-k-1}(S^n \setminus A, S^n \setminus (A \cup P_+)) \xrightarrow{\partial_*} \tilde{H}_{n-k-2}(S^n \setminus (A \cup P_+)) \longrightarrow \tilde{H}_{n-k-2}(S^n \setminus A)$$

Because of Theorem 7.2 and $\tilde{H}_q(S^n \setminus A, S^n \setminus (A \cup P_+)) \cong \tilde{H}_q(S^n)$ this becomes

$$0 \xrightarrow{\partial_*} \tilde{H}_{n-k-1}(S^n \setminus (A \cup P_+)) \longrightarrow \mathbf{Z} \longrightarrow 0 \xrightarrow{\partial_*} \tilde{H}_{n-k-2}(S^n \setminus (A \cup P_+)) \longrightarrow 0$$

and therefore $\tilde{H}_{n-k-1}(S^n \setminus (A \cup P_+)) \cong \mathbf{Z}$ and $\tilde{H}_{n-k-2}(S^n \setminus (A \cup P_+)) = 0$. This completes the proof. $\square$

If we compare Lemma 7.1 and Theorem 7.2 to Lemma 7.5 and Theorem 7.6 it is apparent that, if S^n is replaced by $\mathbf{R}^n$, one more group , namely $\tilde{H}_{n-1}$ is different from 0. A geometric interpretation can be given for this.

Consider the case where $B \subset \mathbf{R}^n$ is a k-ball. There exists an $a > 0$ such that $B \subset \{x \in \mathbf{R}^n : |x| < a\}$. Let $f : S^{n-1} \to \mathbf{R}^n$ be given by $f(x) = ax$. Because f is a homeomorphism and $f(S^{n-1})$ is a deformation retract of $\mathbf{R}^n \setminus B$ we conclude that $f_* : \tilde{H}_{n-1}(S^{n-1}) \to \tilde{H}_{n-1}(\mathbf{R}^n \setminus B)$ is an isomorphism and hence $\tilde{H}_{n-1}(\mathbf{R}^n \setminus B) \cong \mathbf{Z}$. This also illustrates how a generator of $\tilde{H}_{n-1}(\mathbf{R}^n \setminus B)$ can be imagined. If we form the one-point compactification of $\mathbf{R}^n$ by adding one single point $\{\infty\}$ this generating cycle becomes a boundary. This explains why $\tilde{H}_{n-1}(S^n \setminus B) = 0$. A similar argument can be applied to $\mathbf{R}^n \setminus S$.

Having Theorem 7.5 and 7.6 at hand we can state the Jordan-Brouwer theorem for $\mathbf{R}^n$:

Theorem 7.7 *Let $S \subset \mathbf{R}^n$ be homeomorphic to S^{n-1}. Then $\mathbf{R}^n \setminus S$ consists of exactly two components. S is the common boundary of these components.*

The first part follows directly from Theorem 7.6 and the proof of the second part proceeds exactly as the proof of Theorem 7.4. The unbounded component of $\mathbf{R}^n \setminus S$ is called the outside of S and the other component is called the inside.

From Theorem 7.3 we can get a very important corollary, the theorem of the invariance of domain, which is also due to Brouwer.

Theorem 7.8 (Invariance of Domain) *Let $U, V \subset S^n$ be homeomorphic subsets of S^n. If U is open then so is V.*

Proof. Let $f : U \to V$ be a homeomorphism. Let $x \in U$ and $\in V$ such that $f(x) = y$. Take a closed neighbourhood N of x in U which is homeomorphic to I^n and $\dot{N} \approx S^{n-1}$. Now $f(N)$ is a closed neighbourhood of y in V and Theorem 7.1 says that $S^n \setminus f(N)$ is connected. But we also know that $S^n \setminus f(\dot{N})$ has two components, by Theorem 7.3. Now

$$S^n \setminus f(\dot{N}) = (S^n \setminus f(N)) \cup (f(N) \setminus f(\dot{N}))$$

is the disjoint union of two nonempty connected sets. Hence $S^n \setminus f(N)$ and $f(N) \setminus f(\dot{N})$ are the components of $S^n \setminus f(\dot{N})$. This implies that both are open in $S^n \setminus f(\dot{N})$ and therefore $f(N) \setminus f(\dot{N})$ is also open in S^n. But $f(N) \setminus f(\dot{N})$ is an open neighbourhood of y which is entirely contained in V. Since y is arbitrary it follows that V is open. $\square$

A similar theorem may be stated for $\mathbf{R}^n$. The proof is the same. We can also express this theorem in a slightly different manner and get:

Corollary 7.9 *Let U and V be arbitrary subsets of S^n ($\mathbf{R}^n$) having a homeomorphism $f : U \to V$. Then f maps interiour points onto interiour points and boundary points onto boundary points.*

For the rest of this section we want to replace S^n or $\mathbf{R}^n$ by an arbitrary topological space X and discuss some more general results concerning separation, known as the Phragmen-Brouwer properties. The key to these properties is the following theorem.

Theorem 7.10 *Let X be a topological space and let $A, B \subset X$ be nonempty, disjoint, closed subsets such that $X \setminus A$ and $X \setminus B$ are arcwise connected. If $\tilde{H}_1(X) = 0$ then $X \setminus (A \cup B)$ is also arcwise connected.*

Proof. Since $A \cap B = \emptyset$ we have

$$\begin{aligned} X &= (X \setminus A) \cup (X \setminus B) \text{ and} \\ X \setminus (A \cup B) &= (X \setminus A) \cap (X \setminus B). \end{aligned}$$

Hence we can form the Mayer-Vietoris sequence and get

$$\tilde{H}_1(X) \longrightarrow \tilde{H}_0(X \setminus (A \cup B)) \longrightarrow \tilde{H}_0(X \setminus A) \oplus \tilde{H}_0(X \setminus B).$$

Because $X \setminus A$ and $X \setminus B$ are arcwise connected we get

$$0 \longrightarrow \tilde{H}_0(X \setminus (A \cup B)) \longrightarrow 0.$$

Hence $\tilde{H}_0(X \setminus (A \cup B)) = 0$ and $X \setminus (A \cup B)$ is arcwise connected. $\square$

Under the additional assumptions that X is an arcwise connected and locally arcwise connected Hausdorff space we can deduce from Theorem 7.10 the Phragmen-Brouwer properties which are listed below. The first property is indeed an immediate corollary of Theorem 7.10.

Theorem 7.11 (Property I) *Let A, B be two nonempty, disjoint subsets of X. If two points x and y belong to both the same component of $X \setminus A$ and $X \setminus B$ they also belong to the same component of $X \setminus (A \cup B)$.*

Theorem 7.12 (Property II) *Let A be a closed, connected, nonempty subset of X. Then each component of $X \setminus A$ has a connected boundary.*

Theorem 7.13 (Property III, Unicoherence) *Let A, B be two closed, connected subsets of X such that $X = A \cup B$. Then $A \cap B$ is connected.*

Theorem 7.14 (Property IV) *Let A be a closed subset of X and let C_1, C_2 be two disjoint components of $X \setminus A$ which have the same boundary B. Then B is connected.*

Theorem 7.15 (Property V) *Let A, B be two disjoint, closed subsets of X and let $x \in A$ and $y \in B$. Then there exists a closed, connected subset $C \subset X \setminus (A \cup B)$ such that x and y belong to different components of $X \setminus C$.*

Using elementary arguments from point set topology it may be shown that all these properties are equivalent. Because $H_1(S^n) = H_1(\mathbf{R}^n) = 0$ for $n \geq 1$ it follows that S^n and $\mathbf{R}^n$ have the Phragmen-Brouwer properties.

8 The Schönflies Theorem

We continue to examine the separation properties of S^n and $\mathbf{R}^n$. Although we shall only speak about S^n from now on everything can be reformulated in terms of $\mathbf{R}^n$ without significant changes. In particular for every theorem about separation of S^n there is a corresponding theorem for $\mathbf{R}^n$.

In the last section we saw that a homeomorph S of S^{n-1} separates S^n into two components. A few more questions might be asked about this. If S were the standard S^{n-1} in S^n then the corresponding components U and V of $S^n \setminus S$ would be the upper and lower hemispheres of S^n. That means that the closures of U and V are homeomorphic to the n-disc D^n. This observation leads to the Schönflies conjecture.

Conjecture 8.1 (Schönflies Conjecture) *Let S be a subset of S^n which is homeomorphic to S^{n-1}. Then the closure of each of the components of $S^n \setminus S$ is homeomorphic to D^n.*

It turns out that the Schönflies conjecture is true for $n = 2$ but does not hold for $n \geq 3$ without additional assumptions. First, we will examine the 2-dimensional case. We state the corresponding result.

Theorem 8.2 (Schönflies Theorem) *Let $S \subset S^2$ be homeomorphic to S^1. Then the closures of the components of $S^2 \setminus S$ are homeomorphic to D^2.*

We may also express this differently as

Theorem 8.3 (Schönflies Theorem, Second Form) *Let $S \subset S^2$ be homeomorphic to S^1. Then the homeomorphism of $S \subset S^2$ to $S^1 \subset S^2$ can be extended to give a homeomorphism of S^2 onto S^2.*

We will not give the proof which is quite complicated. The reader is referred to Christenson/Voxman or Moise. Here we will prove the Schönflies theorem for polygons only. For this we have to come back to the concept of a simplicial complex as established in Section 0.

Let S be a polygon in S^2. We choose one component U of $S^2 \setminus S$. It may be shown that $\overline{U}$ can be triangulated, i.e. $\overline{U}$ is a finite complex $|K|$. We call a 2-simplex $\sigma \in K$ a *free simplex* if $\sigma \cap S$ consists of one or two edges of σ. For our purpose we need a lemma.

Lemma 8.4 *Let S be a polygon in S^2 and let K be a triangulation of the closure of a component U of $S^2 \setminus S$. If K has more than one 2-simplex than K has a free 2-simplex.*

Proof. We will prove the stronger result that K has at least two free 2-simplexes. The proof is by induction on the number of 2-simplexes of K. If K has exactly two 2-simplexes then both are free.

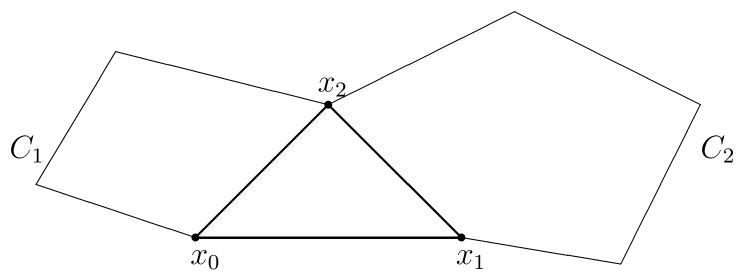

Figure 1

Assume now that K has more than two 2-simplexes. There are two 2-simplexes σ, τ of K with $\sigma \cap S$ and $\tau \cap S$ consisting of at least one edge of σ and τ, respectively. If both σ and τ are free then there is nothing to prove. Suppose that

$$\sigma = x_0x_1x_2 \in K$$

is not free. Let $x_0x_1 \subset \sigma \cap S$ as in Figure 1. It follows that also $x_2 \in \sigma \cap S$. Therefore S can be decomposed into two broken lines C_1 and C_2 by the points x_0 and x_2. Let U_1 and U_2 be the interiour of $C_1 \cup x_0x_2$ and $C_2 \cup x_0x_2$, respectively. Then $|K| = \overline{U}_1 \cup \overline{U}_2$. Let K_1 be the complex consisting of all simplexes of K that lie in $\overline{U}_1$, together with σ and its faces, and let K_2 be the complex consisting of all simplexes of K that lie in $\overline{U}_2$, which also contains σ. By the inductive hypothesis K_1 and K_2 have two free 2-simplexes each and hence both K_1 and K_2 contain one free 2-simplex different from σ. These two simplexes are also free in K which completes the inductive step. $\square$

We can now formulate Theorem 8.2 for polygons.

Theorem 8.5 *Let S be a polygon in S^2. Then the closures of the components of $S^2 \setminus S$ are homeomorphic to D^2.*

Proof. Let U be one of the components of $S^2 \setminus S$. We will construct a homeomorphism $f : S^2 \to S^2$ such that $f(\overline{U})$ is a 2-simplex. Thus $\overline{U}$ is homeomorphic to D^2. Application of the same procedure to the other component gives the result.

Let K_0 be a triangulation of $\overline{U}$ with k 2-simplexes. We will describe a homeomorphism g_1 which reduces the number of free simplexes of K_0 by 1. Hence $K_1 = g_1(K_0)$ is a complex with one less 2-simplex, which still has at least one free 2-simplex by Lemma 8.4. By induction we obtain a complex K_{k-1} which consists of only one 2-simplex and get $f = g_{k-1} \dots g_1$.

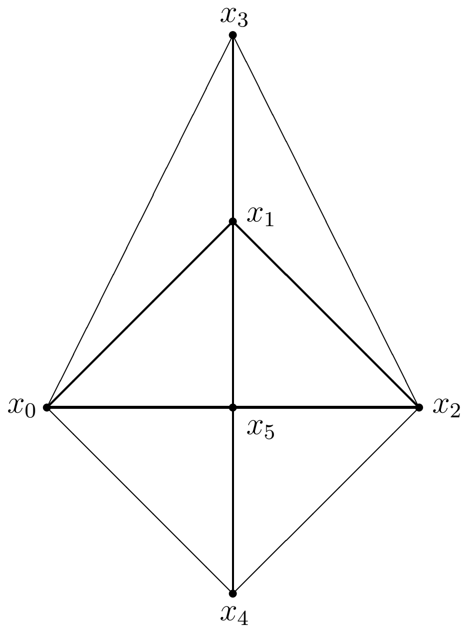

Figure 2

Now we construct g_1. Let $\sigma = x_0x_1x_2$ be a free 2-simplex of K_0. Assume that $\sigma \cap S = x_0x_1$. We choose x_3 and x_4 as in Figure 2 in such a way that the entire figure intersects S only in x_0x_2. Define g_1 to be the identity outside Figure 2. Hence g_1 is the identity on x_0, x_2, x_3 and x_4. Inside Figure 2 let g_1 be the linear map induced by $g_1(x_5) = x_1$.

If $\sigma \cap S = x_0x_1 \cup x_1x_2$ let g_1 be the inverse of the homeomorphism just defined. $\square$

We will now examine the Schönflies conjecture in dimension three. We have an equivalent result to Theorem 8.5.

Theorem 8.6 *Let S be a polyhedral 2-dimensional sphere in S^3. Then the closures of $S^3 \setminus S$ are homeomorphic to D^3.*

The proof of even a slightly stronger form of Theorem 8.6 can be found in the book by Moise.

As mentioned before we do not have an analogon of the 2-dimensional Schönflies theorem for arbitrary curves (Theorem 8.2) in S^3. In fact we can construct some counterexamples. The first one is the Alexander horned sphere (considered in $\mathbf{R}^3$). The horned sphere S is homeomorphic to S^2 but we cannot find a homeomorphism $f : \mathbf{R}^3 \to \mathbf{R}^3$ such that $f(S) = S^2$. From the picture one can already "see" that $\mathbf{R}^3 \setminus S$ is not simply connected because S contains a Cantor set. On the other hand $\mathbf{R}^3 \setminus S^2 = f(\mathbf{R}^3 \setminus S)$ is simply connected and hence such a homeomorphism f can not exist.

Another also intuitive explanation is that it is impossible to form a "membrane" with R as its boundary that does not intersect S. If a homeomorphism $f : \mathbf{R}^3 \to \mathbf{R}^3$ with $f(S) = S^2$ existed then $f(R)$ would have a membrane outside S^2, which gives a contradiction.

Alexander's horned sphere can be modified to obtain homeomorphs of S^{n-1} in $\mathbf{R}^n$ for $n > 3$ which show that the Schönflies conjecture is also false for these dimensions.

The second example which we will describe is due to Antoine. It is called Antoine's necklace. Let T be a solid torus with l solid tori $T_1, \ldots, T_l$ embedded and particularly linked in L (see the book by Moise for a graphical representation). In every T_i we embed l solid tori in exactly the same way that the T_i are embedded in T. After k steps of embeddings we get l^k tori, whose union we denote by A_k. Antoine's necklace A is then defined to be

$$A = \bigcap_{k=1}^{\infty} A_k.$$

The intersection is nonempty. Because the tori in A_k have a small diameter for large k the set A consists only of single points. Since the tori in A_k are also close to each other for large k every point of A is a limit point of A. Since A is also compact it is a Cantor set.

It may further be shown that there exists a set S homeomorphic to S^2 such that $A \subset S \subset T^\circ$. This set S is then again an example of a 2-sphere for which we cannot find a homeomorphism $f : \mathbf{R}^3 \to \mathbf{R}^3$ such that $f(S) = S^2$. For details on this example the reader should consult the book by Moise.

The Schönflies theorem and possible generalizations can also be put into a more general context. Let (X, A) and (X', A') be pairs of spaces. We consider homeomorphisms $f : X \to X'$ such that $f(A) = A'$. We call such a homeomorphism f a homeomorphism between the pairs (X, A) and (X', A'). If we let $X = X'$ we say that A and A' are equivalent subspaces of X if there is a homeomorphism between the pairs (X, A) and (X, A'). We shall now consider the case $X = S^n$ (or $X = \mathbf{R}^n$) and $A \approx S^k$, $k < n$.

From Theorem 8.3 we conclude that if S_1 and S_2 are different embeddings of S^1 into S^2 there is a homeomorphism f of S^2 onto S^2 such that $f(S_1) = S_2$. Thus all homeomorphs of S^1 in S^2 are equivalent. For $\mathbf{R}^2$ this means that there are no knots in the plane. In $\mathbf{R}^3$, however, it is a well known fact that different knots, i.e. nonequivalent embeddings of S^1 into $\mathbf{R}^3$, exist and hence we can not hope to generalize the Schönflies theorem in this direction. We also saw already that there are nonequivalent embeddings of S^2 in S^3. To obtain some more positive results we have to make additional assumptions on the nature of the embedding of S^k into S^n. The first condition is that of tameness.

Definition 8.7 *Let S be homeomorphic to S^k. Then S is said to be a tame imbedding of S^k into S^n if each point $x \in S$ has a neighbourhood (N_1, N_2) in (S^n, S) such that (N_1, N_2) is homeomorphic to $(\mathbf{R}^n, \mathbf{R}^k)$. Otherwise S is called wild.*

Using triangulations this definition might be expressed as

Definition 8.8 *Let S be a triangulable subspace of S^n. If there is a homeomorphism $f : S^n \to S^n$ such that $f(S)$ is a polyhedron then S is tame. Otherwise, as before, S is wild.*

Clearly, the Alexander horned sphere and Antoine's necklace are wild sets in $\mathbf{R}^3$. Tameness gives an answer to the question of equivalence for a large class of pairs of spaces. We just state the result.

Theorem 8.9 *For $n - k \geq 3$ all tame embeddings of S^k into S^n are equivalent.*

The case $n - k = 2$ leads us into the very extensive theory of knots which we will not be concerned about here. To ensure equivalence for $k = n - 1$ we need a kind of "global tameness" condition, namely there must exist a bicollar for $S \approx S^{n-1}$ in S^n. The precise definition is

Definition 8.10 *Let $S \subset S^n$ be homeomorphic to S^{n-1}. Then S is bicollared if there is an embedding $f : S^{n-1} \times I \to S^n$ such that $f(S^{n-1} \times \{1/2\}) = S$.*

This enables us to state the generalized Schönflies theorem. A proof can be found in Christenson/Voxmann.

Theorem 8.11 (Generalized Schönflies Theorem) *Let $S \subset S^n$ be homeomorphic to S^{n-1}. If S is bicollared then the closure of each component of $S^n \setminus S$ is homeomorphic to D^n*

A related problem to the Schönflies theorem is the annulus conjecture. An annulus is a space homeomorphic to $S^{n-1} \times I$. Let S_1 and S_2 be two subsets of $\mathbf{R}^n$ both homeomorphic to S^{n-1} such that S_1 is contained in the inner component of $\mathbf{R}^n \setminus S_2$. Denote the inner components of $\mathbf{R}^n \setminus S_1$ and $\mathbf{R}^n \setminus S_2$ by U_1 and U_2, respectively. Let $U = \overline{U_1 \cap U_2}$. Then the annulus conjecture may be formulated as follows

Conjecture 8.13 (Annulus Conjecture) *If S_1 and S_2 are bicollared then U is an annulus.*

This conjecture is proved for $n \neq 4$. For $n = 4$ the result is still unknown.

9 Historical Comments

Considering a circle in the plane it is intuitively clear that the circle devides the plane into two regions, called the interiour and the exteriour. In the late nineteenth century due to the recent development of analysis, however, it was discovered that continous mappings from the circle into the plane could have very "nonintuitive" properties. One example for this is the square-filling Peano curve, which Peano (1858-1932) defined in 1890. This is on the other hand not a simple closed curve, i.e. it is not homeomorphic to the circle. Therefore separation properties in the plane began to gain interest and in 1893 C. Jordan (1838-1922) gave the first proof of the Jordan curve theorem (Theorem 7.3 for $n = 2$) in his book 'Cours d'Ananlyse'. His proof, however, was incomplete because he took the theorem in the case of polygons for granted and also omitted some details in his argument. The polygonal version of the Jordan curve theorem was proved by N.J. Lennes (1874-1951) in 1903 and by O. Veblen (1880-1960) in 1904. It was also Veblen who gave the first complete proof of the Jordan curve theorem for arbitrary curves in 1905. These proofs, however, did not use homology theory but complicated geometric arguments. Simpler proofs were given by L.E.J. Brouwer (1881-1967) in 1910 and by J.W. Alexander (1888-1971) in 1920.

The generalization of the Jordan curve theorem for dimension $n \geq 2$, called the Jordan-Brouwer theorem (Theorem 7.7) may be split into three parts. Let $S \subset \mathbf{R}^n$ be homeomorphic to S^{n-1}. Then we have:

1. $\mathbf{R}^n \setminus S$ has at least two components.
2. S is the boundary of the components of $\mathbf{R}^n \setminus S$
3. $\mathbf{R}^n \setminus S$ has at most two components.

M.H. Lebesgue (1875-1941) published a sketch of a proof for the first part, which is independent of the other two, in 1911. At first Brouwer mistrusted these ideas because he misunderstood Lebesgue's somewhat unclear language. Later he admitted that his methods could indeed be used for a rigorous proof of part one but did not want to complete the proof himself. Because Lebesgue did not publish any more about the subject no complete proof was available before Alexander's paper in 1922.

Parts two and three were proved by Brouwer in two papers in 1912. In the first of these articles he also proved the theorem of the invariance of domain for which he did not use the separation theorem but a "no separation theorem" similar to Lemma 7.1 which also contributed to the proofs of parts two and three of the Jordan-Brouwer theorem. Brouwer was originally concerned about separation in $\mathbf{R}^n$ but also showed that a related result (Theorem 7.4) holds for S^n.

Alexander generalized the Jordan-Brouwer theorem in 1922 by showing the relation between the Betti numbers of a closed set A in S^n and those of $S^n \setminus A$. The relevant result is the Alexander duality theorem.

The Phragmen-Brouwer properties started with L. Phragmen in 1885 who proved that if A is a compact, connected subset of $\mathbf{R}^2$ then the unbounded component of $\mathbf{R}^2 \setminus A$ has a connected boundary. In 1910 Brouwer showed the more general result that in fact any component of $\mathbf{R}^2 \setminus A$ has a connected boundary. Later the other properties were added and $\mathbf{R}^2$ was replaced by more general spaces X. The dependence of the Phragmen-Brouwer properties on the fact that $H_1(X) = 0$ (Theorem 7.10) was first shown by P. Alexandroff and H. Hopf in 1935.

In 1902 A. Schönflies (1853-1928) announced a converse to the Jordan curve theorem: A set of points which devides the plane into two regions is a curve. He explained his results in a series of papers from 1904 to 1906. In the last paper 1906 he also proved the 2-dimensional Schönflies theorem (Theorem 8.2). But his proof contained errors and he also assummed (like Jordan in 1893) the polygonal version (Theorem 8.5) without proof. Theorem 8.5 was first proved by L.D. Ames (1869-1965) and G.A. Bliss (1876-1951) in 1904. Their papers also contained a proof of the polygonal Jordan curve theorem. The proof of the Schönflies theorem (Theorem 8.2) was corrected by Brouwer in 1910 and 1912.

Naturally the question was asked whether the Schönflies theorem could be extended to higher dimensions. This led to the Schönflies conjecture. The first counterexample for the 3-dimensional case was given by L. Antoine in 1921. A second example for wild spheres which is better known than that by Antoine is the horned sphere which Alexander presented in 1924. The horned sphere is picturially easier to imagine but its mathematical properties are harder to explore. A reason for this might be that Antoine was blind.

Alexander also proved the Schönflies theorem for dimension three for polygonal spheres. The generalized version of the Schönflies theorem (Theorem 8.11) was proved by M. Brown (b. 1931) in 1960 after B. Mazur (b. 1937) had given a proof in 1959 with slightly different assumptions.

For a long time the annulus conjecture had been one of the famous open problems of topology until it was proved for $n \neq 4$ by Kirby, Siebenmann and Wall in 1969.

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