Algorithms and Data Structures

Sorting:
Simple Methods and a Lower Bound

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Large-Scale Sorting

- Imagine you are the IT head of a telco-company
- You have 30.000.000 customers each performing ~100 telephone calls per months, each call creating 200 bytes
  - That's 30M*100*12*200=7.200.000.000.000 bytes per year
  - Somewhere in the 200 bytes is information on revenue per call
  - Imagine the data is in one file, one line per call
- At the end of the year, management wants a list of all customers with aggregated revenue per day (for one year)
  - That's ~30M*12*30 ~ 10.000.000.000 real numbers
- Problem: How can we compute these 10E9 numbers?

Approach 0a: Load into Memory and Scan

- This won't work
- Data is too big to be loaded into main memory

Approach 0b: Load into a DBMS and use SQL

- This will work
- Not topic of our lecture

- [Will be slow – inserting is costly]
- [Better to already keep the data in a RDBMS – no loading]
- [DBMS will use the same trick we present right now]

Approach 1: Scan and Keep Intermediate Results

- Eventually, we need 10E9 real numbers
- Scan the file from start to end
  - Build table (list! how?) of every combination of customer and day
  - When reading a record, look-up combination in table and update
- That‶s fast (if the table-look-up is fast)
- But we need ~64GB
- What if want the sum for each day over 10 years?
- This won‶t scale

Approach 2: Partition Data, Multiple Reads

- Assume we can keep 30M*30 ~ 1E9 numbers in memory
  - Solve the problem month-by-month (1 month ~ 30 days)
  - Read the call-file 12 times, each time computing aggregates for all customers and the days of one month
  - This will be slow

1st read	2nd read	3rd read	…
Meier, 10.1.2010	Meier, 10.1.2010	Meier, 10.1.2010	Meier, 10.1.2010
Müller, 18.4.2010	Müller, 18.4.2010	Müller, 18.4.2010	Müller, 18.4.2010
Meier, 1.2.2010	Meier, 1.2.2010	Meier, 1.2.2010	Meier, 1.2.2010
Meier, 18.1.2010	Meier, 18.1.2010	Meier, 18.1.2010	Meier, 18.1.2010
Schmidt, 14.1.2010	Schmidt, 14.1.2010	Schmidt, 14.1.2010	Schmidt, 14.1.2010
Schmidt, 6.4.2010	Schmidt, 6.4.2010	Schmidt, 6.4.2010	Schmidt, 6.4.2010
Müller, 27.2.2010	Müller, 27.2.2010	Müller, 27.2.2010	Müller, 27.2.2010
Müller, 9.4.2010	Müller, 9.4.2010	Müller, 9.4.2010	Müller, 9.4.2010
Schmidt, 1.3.2010	Schmidt, 1.3.2010	Schmidt, 1.3.2010	Schmidt, 1.3.2010
Schmitt, 9.2.2010	Schmitt, 9.2.2010	Schmitt, 9.2.2010	Schmitt, 9.2.2010
Schmitt, 30.3.2010	Schmitt, 30.3.2010	Schmitt, 30.3.2010	Schmitt, 30.3.2010
Schmitt, 3.1.2010	Schmitt, 3.1.2010	Schmitt, 3.1.2010	Schmitt, 3.1.2010
…	…	…	…

Approach 3: Sorting

- Alternative?
  - Sort the file by customer and day
  - Read sorted file once and compute aggregates on the fly
  - Whenever a pair (day, customer) is finished (i.e., new values appear), sum can be written out and next day/customer starts
  - This will be very fast
  - Needs virtually no memory during counting
- But: Can we sort ~3 billion records using less than 12 reads?

Meier, 10.1.2010
Meier, 10.1.2010 → Sum
Meier, 1.2.2010 → Sum
Müller, 27.2.2010 → Sum
Müller, 9.4.2010
Müller, 9.4.2010 → Sum
Schmidt, 14.1.2010 …
Schmidt, 1.3.2010
Schmidt, 6.4.2010
Schmitt, 3.1.2010
Schmitt, 3.1.2010
Schmitt, 30.3.2010
…

Content of this Lecture

- Sorting
- Simple Methods
- Lower Bound

Sorting

- Assumptions
  - We have n values (integer, called keys) that should be sorted
  - Values are stored in an array S (i.e., O(1) access to i'th element)
    - Sorting in other list implementations is very different
  - Comparing two values costs O(1)
  - We usually count # of comparisons; sometimes also # of swaps
  - Values are not interpreted
    - We do not know what a "big" value is or how many percent of all values are smaller than a given value or …
  - All we can do is compare two values
- We seek a permutation π of the indexes of S such that

$$\forall i,j \le n \text{ with } \pi(i)<\pi(j) : S[\pi(i)] \le S[\pi(j)]$$

Variations

- External versus internal sorting
  - Internal sorting: S fits into main memory
  - External sorting: There are too many records to fit in memory
  - We only look at internal sorting (see DB lecture)
- In-place or with additional memory
  - In-place sorting only requires a constant (independent of n) amount of additional memory (on top of S)
  - We will look at both
- Pre-Sorting
  - Some algorithms can take advantage of an existing (incomplete, erroneous) order in the data, some not
  - We will not exploit pre-sorting

Applications

- Sorting is a ubiquitous task in computer science
  - [OW93] claims that 25% of all computing time is spent in sorting
- Second example: Information Retrieval
  - Imagine you want to build g*****++
  - Fundamental operation: In a very large set of documents, find those that contain a given set of keywords
    - [Note: That's not what a search engine does in reality!]
  - Popular way of doing this: Build an inverted index

Inverted Index

ID	Text
1	Baseball is played during summer months.
2	Summer is the time for picnics here.
3	Months later we found out why.
4	Why is summer so hot here?

Term	Freq	Document ids
baseball	1	[1]
during	1	[1]
found	1	[3]
here	2	[2], [4]
hot	1	[4]
is	3	[1], [2], [4]
months	2	[1], [3]
summer	3	[1], [2], [4]
the	1	[2]
why	2	[3], [4]

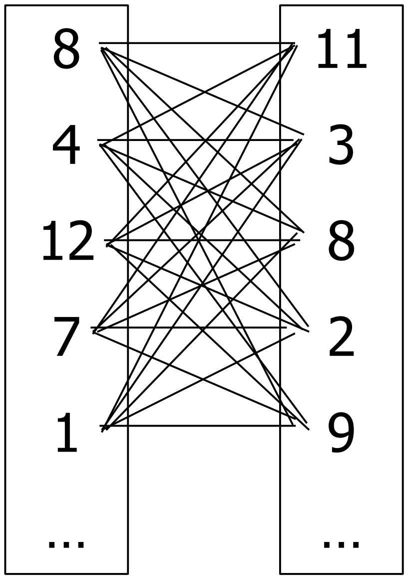
Source: http://docs.lucidworks.com

Answering a IR-style Query

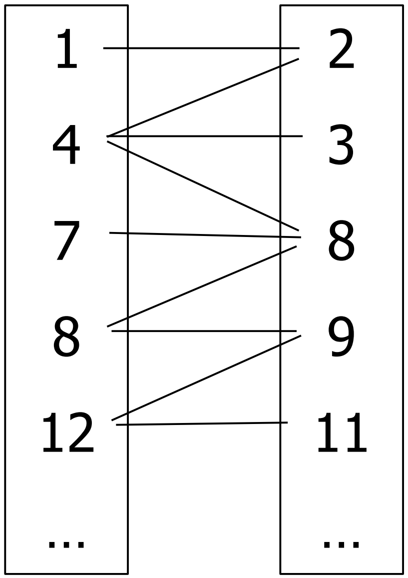
- A query is a set of keywords
- Finding the answer
  - For each keyword k_i of the query, find list d_i of docs containing k_i from inverted index
  - Build intersection of all d_i
  - Docs in this list are your answer
- Imagine the query "the man eats a bread" on the Web
  - Doc-list for "the" and "a" will contain >10 billion documents
- How do we compute the intersection of two sets of 10 billion IDs?

Intersection of Two Sets

With non-sorted sets:
O(m*n)



With sorted sets:
O(n+m)



Content of this Lecture

- Sorting
- Simple Methods
  - Selection sort
  - Insertion sort
  - Bubble sort
- Lower Bound

Recall: Selection Sort

```
S: array_of_names;
n := |S|
for i = 1..n-1 do
  for j = i+1..n do
    if S[i]>S[j] then
      tmp := S[j];
      S[j] := S[i];
      S[i] := tmp;
    end if;
  end for;
end for;
```

- Analysis showed that selection sort is in $O(n^2)$
- It is easy to see that selection sort also is in $\Omega(n^2)$
- How often do we swap values?
  - That depends a lot on the pre-sorted'ness of the array
  - But actually we can do a bit better

Selection Sort Improved

```
S: array_of_names;
n := |S|
for i = 1..n-1 do
  min_pos := i;
  for j = i+1..n do
    if S[min_pos]>S[j] then
      min_pos := j;
    end if;
  end for;
  if min_pos != i then
    tmp := S[i];
    S[i] := S[min_pos];
    S[min_pos] := tmp;
  end if;
end for;
```

- How often do we swap values?
  - Once for every position
  - Thus: O(n) swaps
  - But more (cheap) assignments

Analogy

- Let's assume you keep your cards sorted
- How to get this order?
  - Selection sort: Take up all cards at once and build sorted prefixes of increasing length
  - Insertion sort: Take up cards one by one and sort every new card into the sorted subset in your hand
  - Bubble sort: Take up all cards at once and swap neighbors until everything is fine

Insertion Sort

```
S: array_of_names;
n := |S|
for i = 2..n do
  j := i;
  key := S[j];
  while (S[j-1]>key) and (j>1) do
    S[j] := S[j-1];
    j := j-1;
  end while;
  S[j] := key;
end for;
```

- After each loop of i, the prefix S[1..i] of S is sorted
- While-loop runs backwards from current position (to be inserted) until value gets smaller than S[j]
- Example: 5 4 8 1 6
- One problem is the required movement of many values until correct place is found
  - Could be implemented much better with a double-linked list

Complexity (Worst Case)

```
S: array_of_names;
n := |S|
for i = 2..n do
  j := i;
  key := S[j];
  while (S[j-1]>key) and (j>1) do
    S[j] := S[j-1];
    j := j-1;
  end while;
  S[j] := key;
end for;
```

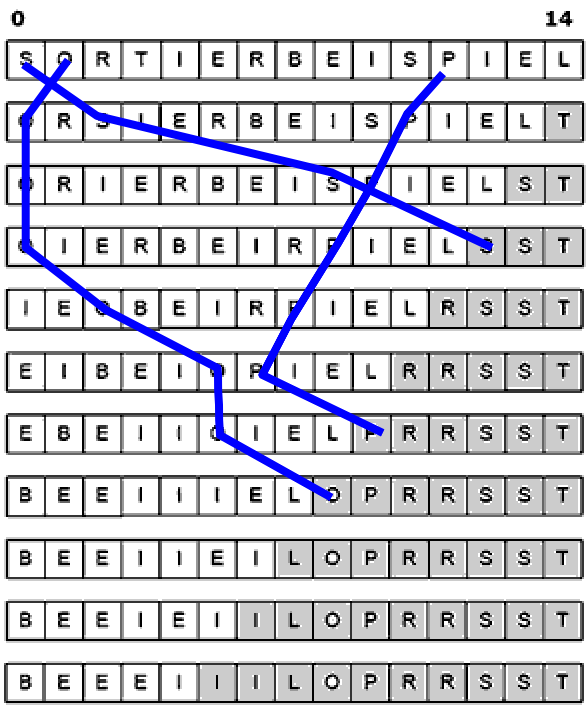
- Comparisons
  - Outer loop: n times
  - Inner-loop: i times
  - Thus, $O(n^2)$
- How many swaps?
  - (We move and don't swap, but both are in O(1))
  - In worst-case, every comparison incurs a swap
  - Thus: $O(n^2)$
- We got worse?

Complexity (Best Case)

```
S: array_of_names;
n := |S|
for i = 2..n do
  j := i;
  key := S[j];
  while (S[j-1]>key) and (j>1) do
    S[j] := S[j-1];
    j := j-1;
  end while;
  S[j] := key;
end for;
```

- Assume the best case: S is already sorted
- Comparisons
  - Outer loop: n times
  - Inner-loop: 1 time
  - Thus, O(n)
- Swaps
  - None
- We might be better!

Bubble Sort



Source: HKI, Köln

- Go through array again and again
- Compare all direct neighbors
- Swap if in wrong order
- Repeat until a loop finishes without a single swaps
- Analysis: About as good/bad as the others (so far)
  - Worst case $O(n^2)$ comparisons and $O(n^2)$ swaps
  - Best case $O(n)$ comparisons and 0 moves / swaps

Summary

	Comparisons worst case	Comparisons best case	Additional space	Swaps/moves worst/best
Selection Sort	$O(n^2)$	$O(n^2)$	$O(1)$	$O(n)$
Insertion Sort	$O(n^2)$	$O(n)$	$O(1)$	$O(n^2)$ / $O(n)$
Bubble Sort	$O(n^2)$	$O(n)$	$O(1)$	$O(n^2)$ / $O(1)$

Summary

	Comparisons worst case	Comparisons best case	Additional space	Swaps/moves worst/best
Selection Sort	$O(n^2)$	$O(n^2)$	$O(1)$	$O(n)$
Insertion Sort	$O(n^2)$	$O(n)$	$O(1)$	$O(n^2)$ / $O(n)$
Bubble Sort	$O(n^2)$	$O(n)$	$O(1)$	$O(n^2)$ / $O(1)$
Merge Sort	$O(n*\log(n))$	$O(n*\log(n))$	$O(n)$	$O(n*\log(n))$

Summary

	Comparisons worst case	Comparisons best case	Additional space	Moves worst/best
Selection Sort	$O(n^2)$	$O(n^2)$	$O(1)$	$O(n)^*$
Insertion Sort	$O(n^2)$	$O(n)$	$O(1)$	$O(n^2)$ / $O(n)$
Bubble Sort	$O(n^2)$	$O(n)$	$O(1)$	$O(n^2)$ / $O(1)$
Merge Sort	$O(n*log(n))$	$O(n*log(n))$	$O(n)$	$O(n*log(n))$
Magic Sort (?)	$O(n)$			$O(n)$

Content of this Lecture

- Sorting
- Simple Methods
- Lower Bound

Lower Bound

- We found three algorithms with WC-complexity $O(n^2)$
- Maybe there is no better algorithm?
- There are some in $O(n*\log(n))$
- Maybe there are even better algorithms?

- Is there a lower bound on the number of comparisons?

Lemma

- Lemma
  To sort a list of n distinct keys using only key comparisons, every algorithm needs $\Omega(n*log(n))$ comp's in worst case

- Implications
  - We cannot sort with less than O(n*log(n)) comparisons in worst case
  - Still, different algorithms with O(n*log(n)) may exhibit different real runtimes
  - We can be better, when other operations than comparisons are allowed – see radix sort

Proof Structure

- We find the best safe way to find the right permutation π
- There are n! different permutations
- Each could be the right one
  - And there is only one "right one"
- To find the right one, we may only compare two keys
- Every comparison splits the group of all permutations into two disjoint partitions
  - One with all permutations where the result of the test is TRUE
  - One with all permutations where the result of the test is FALSE
- How often do we need to compare at least until every partition has size 1
  - At least: In the best of all worlds

Decision Tree

1	8	6	3	5	9	3	1	7
5	3	7	1	8	3	6	7	1
9	6	1	5	3	2	4	8	6
4	4	3	6	1	6	8	3	2
7	2	5	8	4	5	9	2	5
2	7	4	9	9	8	2	9	9
3	1	8	4	7	7	1	5	4
6	5	9	1	1	4	7	4	5
8	9	5	2	6	1	5	3	3

Some exemplary permutations (columns) of an arbitrary list S with |S|=9

Example

`S[5]<S[7]?`

1	8	6	3
5	3	7	1
9	6	1	5
4	4	3	6
3	1	5	4
2	7	4	9
7	2	8	8
6	5	9	1
8	9	5	2

5	9	3	1	7
8	3	6	7	1
3	2	4	8	6
1	6	8	3	2
7	7	9	5	5
9	8	2	9	9
4	5	1	2	4
1	4	7	4	5
6	1	5	3	3

General Case

$S[i_1]<S[j_1]?$

1	8	6	3
5	3	7	1
9	6	1	5
4	4	3	6
7	2	5	8
2	7	4	9
3	1	8	4
6	5	9	1
8	9	5	2

5	9	3	1	7
8	3	6	7	1
3	2	4	8	6
1	6	8	3	2
4	5	9	2	5
9	8	2	9	9
7	7	1	5	4
1	4	7	4	5
6	1	5	3	3

All permutations of S where the value at position i_1 is smaller than the value at position j_1

All permutations of S where the value at position i_1 is larger than the value at position j_1

Decision Tree

$S[i_1]<S[j_1]?$

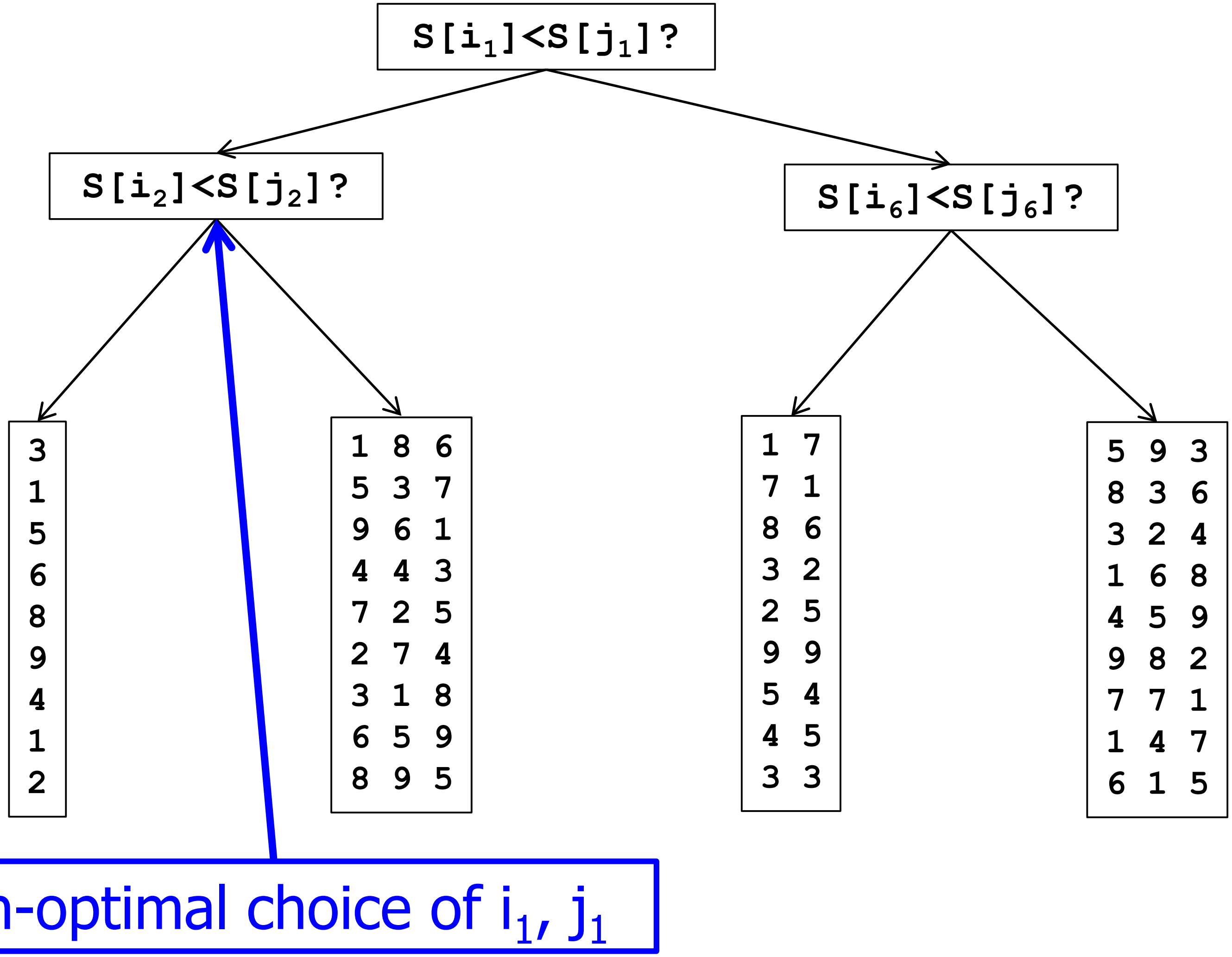
1	8	6	3
5	3	7	1
9	6	1	5
4	4	3	6
7	2	5	8
2	7	4	9
3	1	8	4
6	5	9	1
8	9	5	2

5	9	3	1	7
8	3	6	7	1
3	2	4	8	6
1	6	8	3	2
4	5	9	2	5
9	8	2	9	9
7	7	1	5	4
1	4	7	4	5
6	1	5	3	3

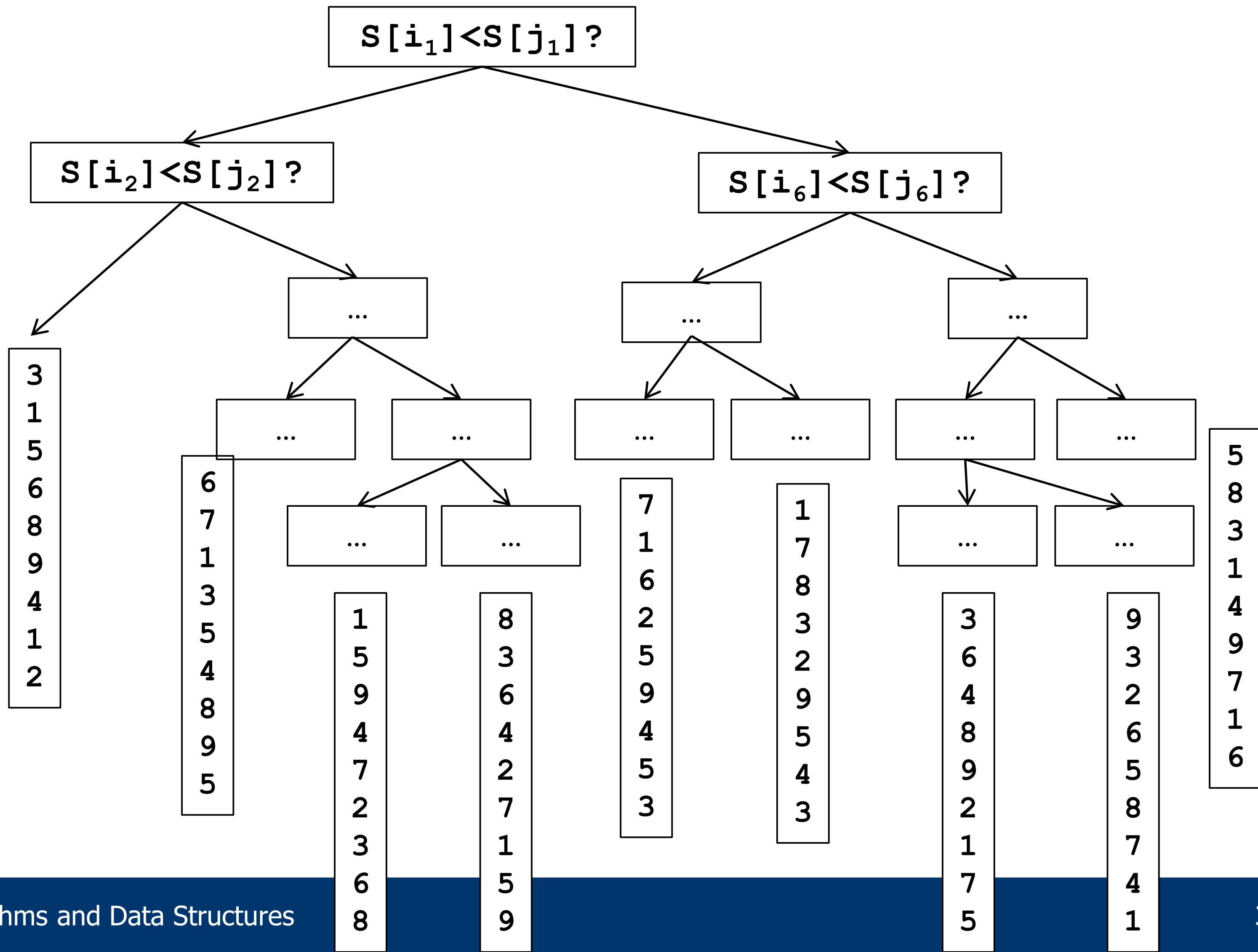
$S[i_2]<S[j_2]?$

$S[i_6]<S[j_6]?$

Decision Tree



Full Decision Tree

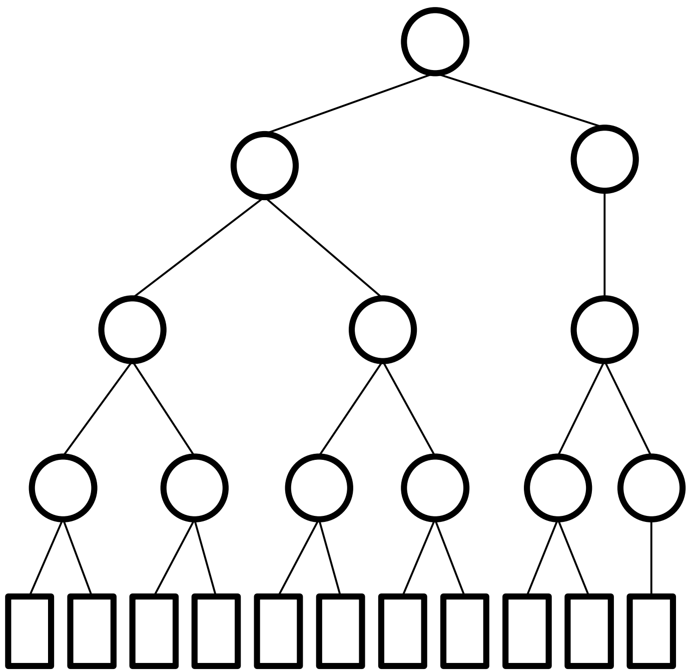


Optimal Sequence of Comparisons

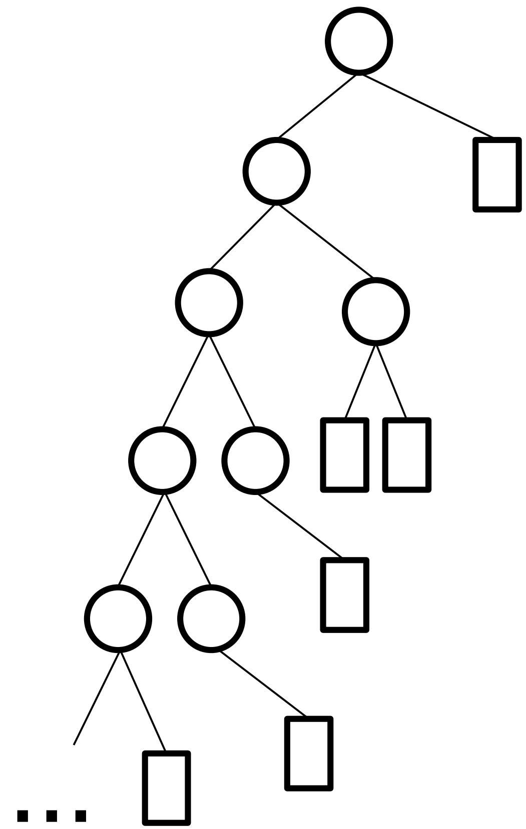
- We have no clue about which concrete series of comparisons is optimal for a given list
- But: Here we are looking for a lower bound: We may always assume to take the best choice
- Best choice: Creating only 1-partitions with as few comparisons as possible
- If we always magically take the best choice – how long can we still need?
- Thus, we want to know the length of the longest path through the optimal (lowest) decision tree
  - Even in the best of all worlds we may need to make this number of comparisons to find the correct permutation
- The optimal tree is the one with the shortest longest path

Intuition

Good
(not optimal)



Bad

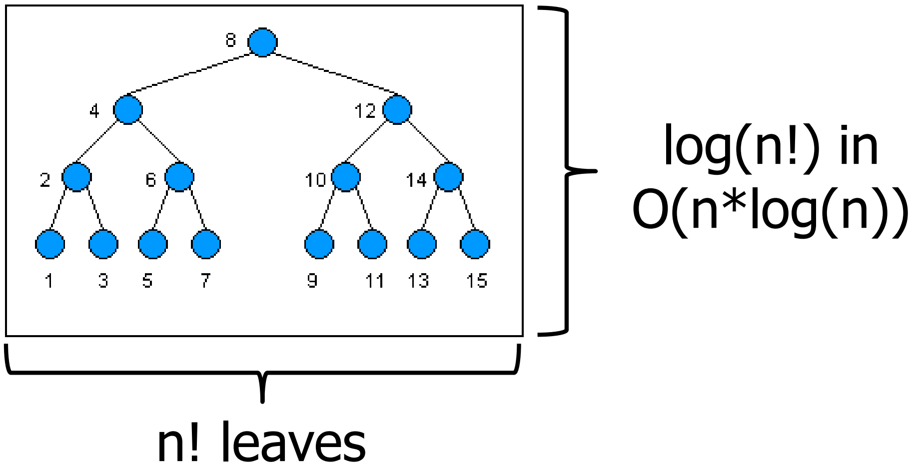


Shortest Longest Path

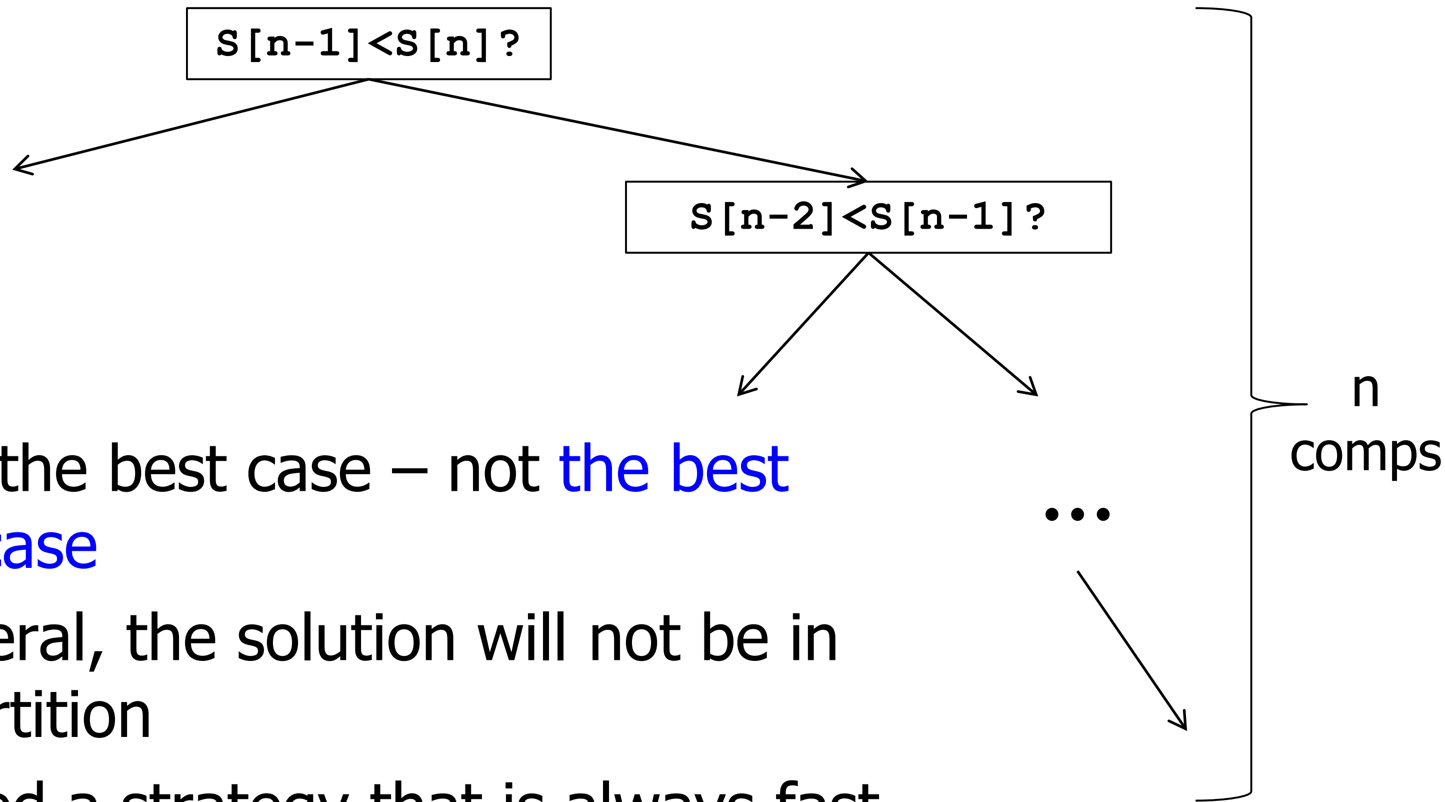
- Definition
  The height of a binary tree is the length of its longest path.
- Lemma
  A binary tree with k leaves has at least height log(k).
- Proof
  - Every inner node has at most two children
  - To cover as many leaves as possible in the level above the leaves, we need ceil(k/2) nodes
  - In the second-last level, we need ceil(k/2/2) nodes
  - Etc.
  - After log(k) levels, only one node remains (root)
  - qed.

Putting it all together

- Our decision tree has n! leaves
- The height of a binary tree with n! leaves is at least log(n!)
- Thus, the longest path in the optimal tree has at least log(n!) comparisons
- Since $n! \geq (n/2)^{n/2}$: $\log(n!) \geq \log((n/2)^{n/2}) = n/2 * \log(n/2)$
- This gives the overall lower bound $\Omega(n*\log(n))$
- qed.



Stop: Why not test in O(n)?



- This is the best case – not the best worst case
- In general, the solution will not be in this partition
- We need a strategy that is always fast, not "faster" in some cases

Exemplary Exam Questions

- Give best case and worst case instances for the following algorithms: insertion sort, bubble sort. Explain your examples
- Proof that bubble sort is in O(n2) and $\Omega(n^2)$ worst case (comparisons)
- Image a list S consisting of k sorted subarrays of arbitrary size (example for k=4: <1,6,7,8,2,5,1,5,7,9,3,5>). Find an algorithm for sorting S which runs in $O(n*k)$